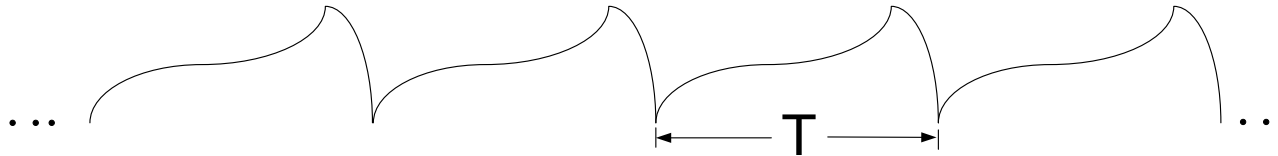


EE 505

Lecture 5

Spectral Characterization

Spectral Analysis



If $f(t)$ is periodic

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \theta_k)$$

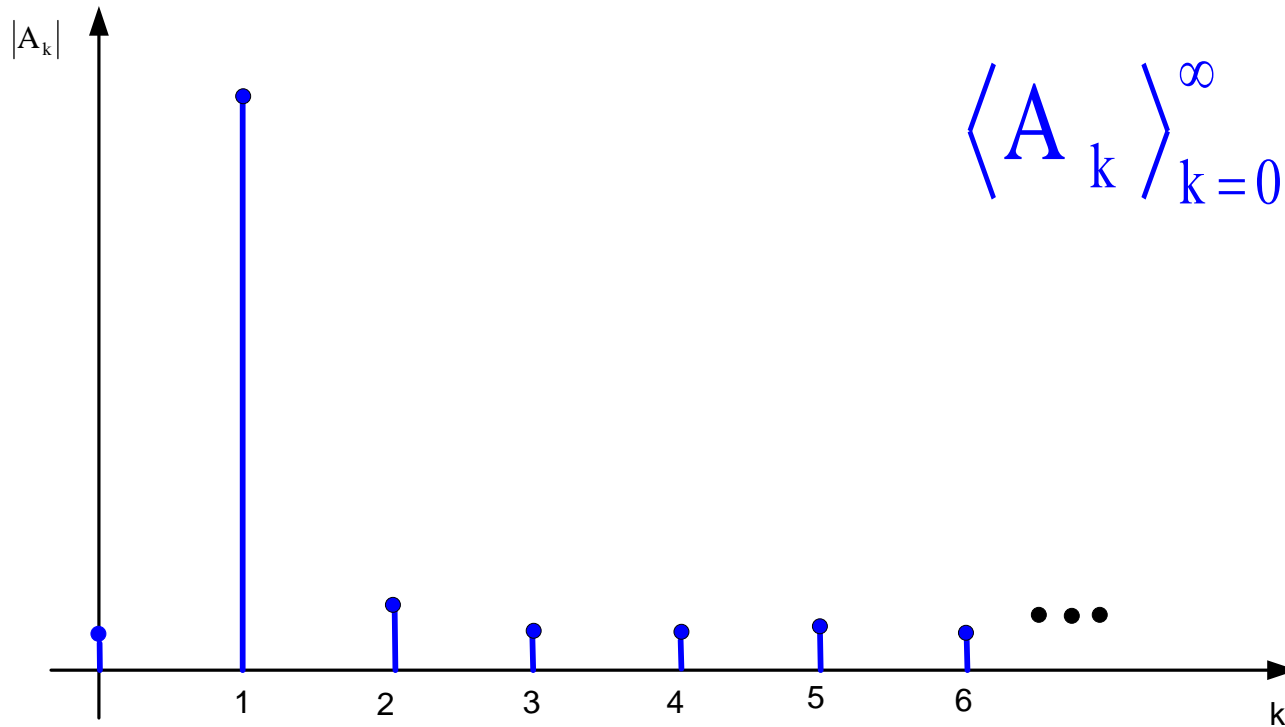
alternately

$$f(t) = A_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega t) + \sum_{k=1}^{\infty} b_k \cos(k\omega t) \quad \omega = \frac{2\pi}{T}$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

Termed the Fourier Series Representation of $f(t)$

Distortion Analysis



For a low distortion signal, the 2nd and higher harmonics are generally much smaller than the fundamental

The magnitude of the harmonics generally decrease rapidly with k for low distortion signals

Distortion Analysis

Total Harmonic Distortion, THD

$$\text{THD} = \frac{\text{RMS voltage in harmonics}}{\text{RMS voltage of fundamental}}$$

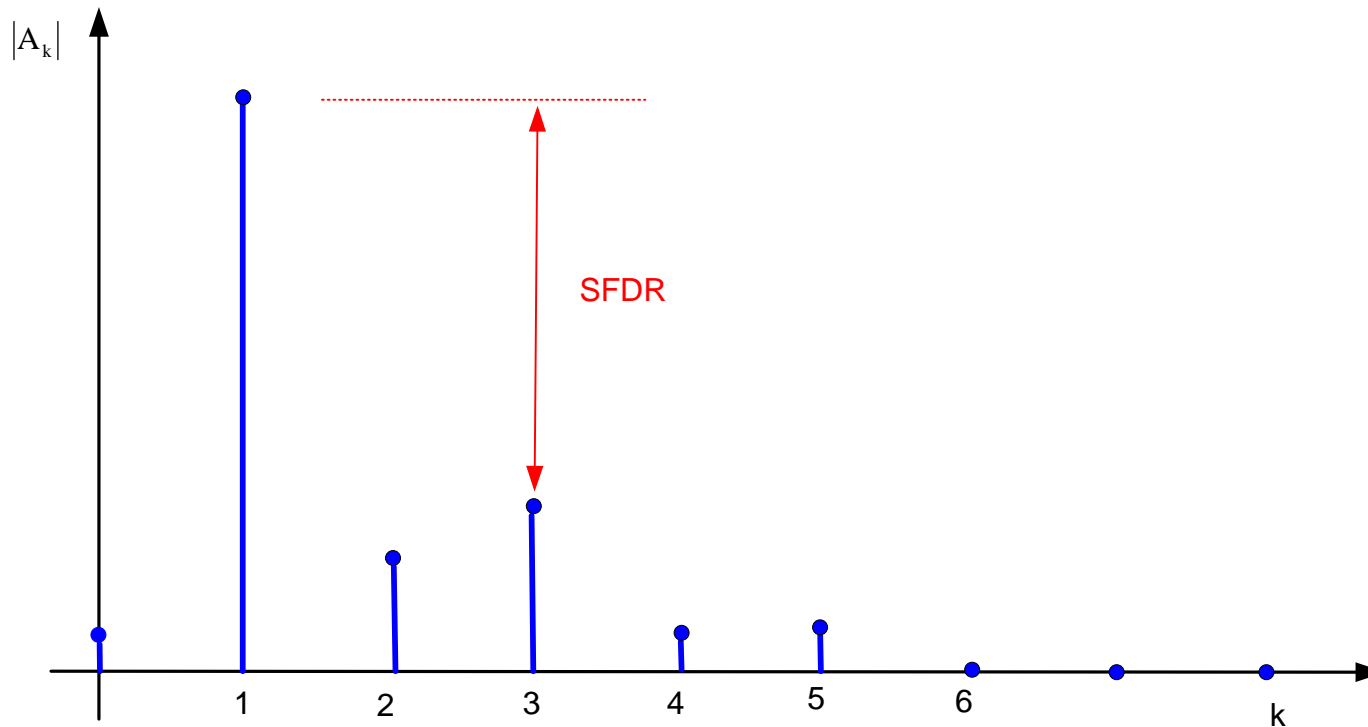
$$\text{THD} = \frac{\sqrt{\left(\frac{A_2}{\sqrt{2}}\right)^2 + \left(\frac{A_3}{\sqrt{2}}\right)^2 + \left(\frac{A_4}{\sqrt{2}}\right)^2 + \dots}}{\frac{A_1}{\sqrt{2}}}$$

$$\text{THD} = \frac{\sqrt{\sum_{k=2}^{\infty} A_k^2}}{A_1}$$

Distortion Analysis

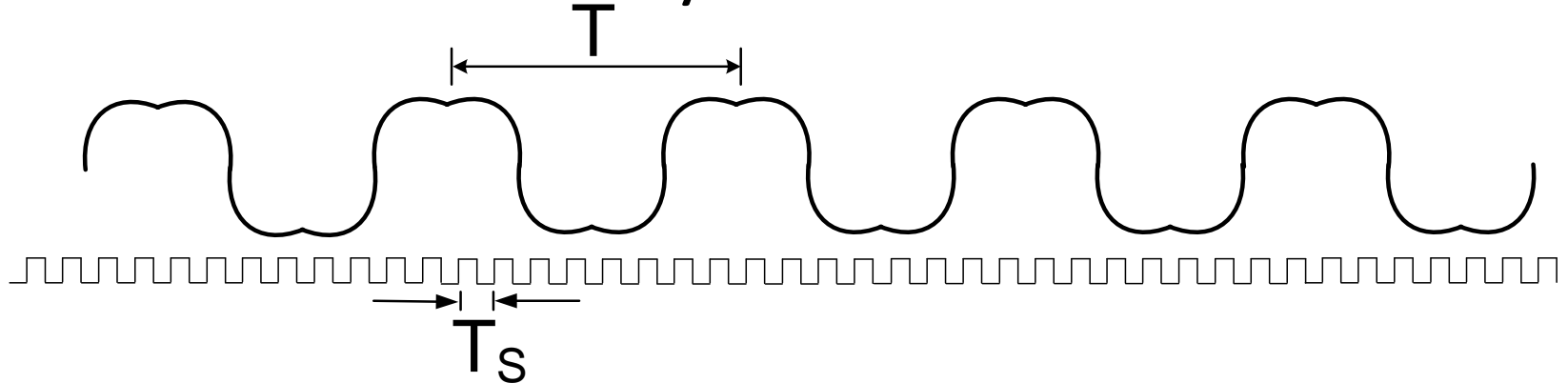
Spurious Free Dynamic Range, SFDR

The SFDR is the difference between the fundamental and the largest harmonic



SFDR is usually determined by either the second or third harmonic

Distortion Analysis



NOTATION:

T: Period of Excitation

T_s : Sampling Period

N_p : Number of periods over which samples are taken

N: Total number of samples

$$N_p = \frac{NT_s}{T}$$

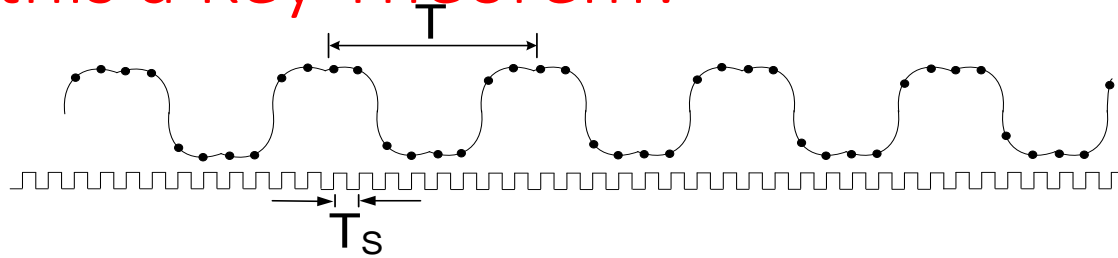
Note: N_p is not an integer unless a specific relationship exists between N, T_s and T

$$h = \text{Int}\left(\left[\frac{N}{2}-1\right]\frac{1}{N_p}\right)$$

Note: The function $\text{Int}(x)$ is the integer part of x

Review from Last Lecture

Why is this a Key Theorem?



THEOREM: If N_p is an integer and the periodic signal $x(t)$ is band limited to f_{MAX} , then

$$|A_m| = \frac{2}{N} |X(mN_p + 1)| \quad 0 \leq m \leq h - 1$$

and $X(k) = 0$ for all k not defined above

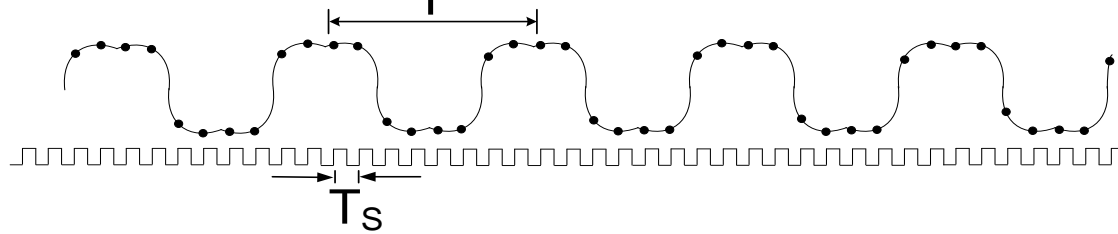
where $\langle X(k) \rangle_{k=0}^{N-1}$ is the DFT of the sequence $\langle x(kT_s) \rangle_{k=0}^{N-1}$

$$f = 1/T, \quad f_{MAX} = \frac{f}{2} \cdot \left\lceil \frac{N}{N_p} \right\rceil, \text{ and } h = \text{Int} \left(\left\lceil \frac{N}{2} - 1 \right\rceil \frac{1}{N_p} \right)$$

- DFT requires dramatically less computation time than the integrals for obtaining Fourier Series coefficients
- Can easily determine the sampling rate (often termed the Nyquist rate) to satisfy the band limited part of the theorem

Review from Last Lecture

How is this theorem abused?



THEOREM: If N_p is an integer and the periodic signal $x(t)$ is band limited to f_{MAX} , then

$$|A_m| = \frac{2}{N} |X(mN_p + 1)| \quad 0 \leq m \leq h - 1$$

and $X(k) = 0$ for all k not defined above

where $\langle X(k) \rangle_{k=0}^{N-1}$ is the DFT of the sequence $\langle x(kT_s) \rangle_{k=0}^{N-1}$

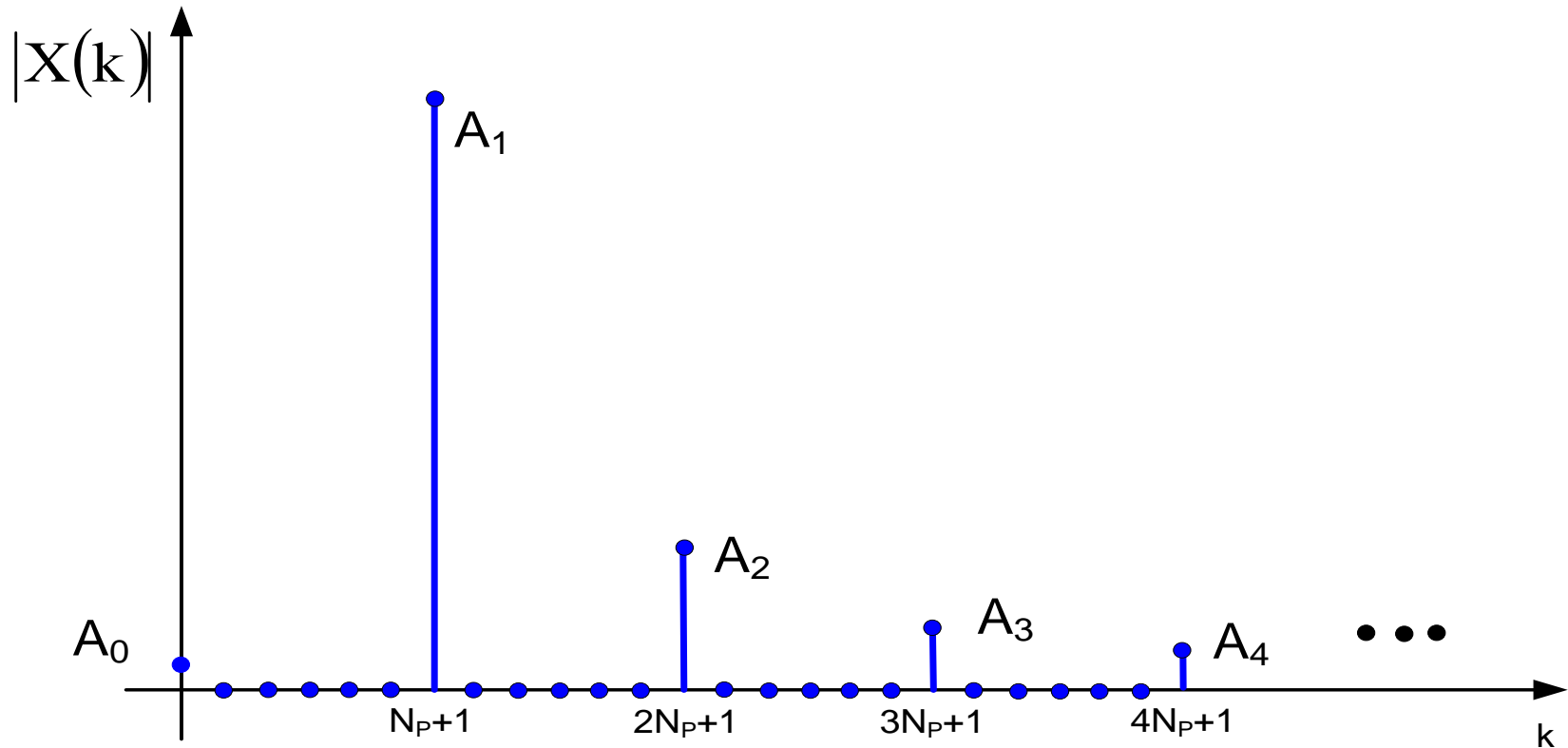
$$f = 1/T, \quad f_{MAX} = \frac{f}{2} \cdot \left\lceil \frac{N}{N_p} \right\rceil, \text{ and } h = \text{Int} \left(\left\lceil \frac{N}{2} - 1 \right\rceil \frac{1}{N_p} \right)$$

- Much evidence of engineers attempting to use the theorem when N_p is not an integer
- Challenging to have N_p an integer in practical applications
- Dramatic errors can result if there are not exactly an integer number of periods in the sampling window

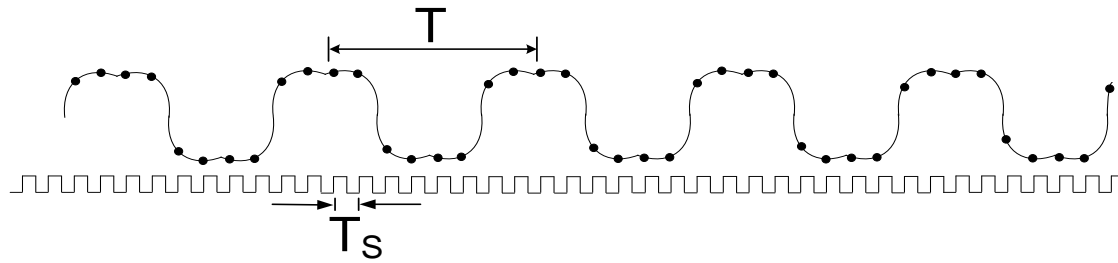
Review from Last Lecture

Distortion Analysis

If the hypothesis of the theorem are satisfied, we thus have



FFT is a computationally efficient way of calculating the DFT, particularly when N is a power of 2



THEOREM: If N_p is an integer and the periodic signal $x(t)$ is band limited to f_{MAX} , then

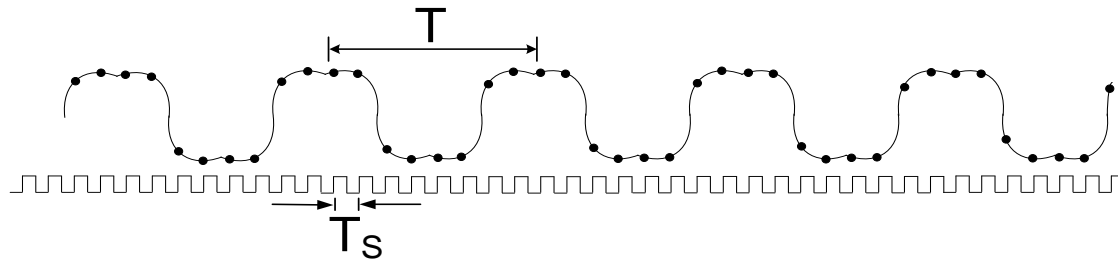
$$|A_m| = \frac{2}{N} |X(mN_p + 1)| \quad 0 \leq m \leq h - 1$$

and $X(k) = 0$ for all k not defined above

where $\langle X(k) \rangle_{k=0}^{N-1}$ is the DFT of the sequence $\langle x(kT_s) \rangle_{k=0}^{N-1}$

$$f = 1/T, \quad f_{MAX} = \frac{f}{2} \cdot \left\lceil \frac{N}{N_p} \right\rceil, \text{ and } h = \text{Int} \left(\left\lceil \frac{N}{2} - 1 \right\rceil \frac{1}{N_p} \right)$$

Note: Band limited part of the hypothesis is equivalent to stating that the sampling rate is greater than the Nyquist rate of $x(t)$



THEOREM: If N_p is an integer and the periodic signal $x(t)$ is band limited to f_{MAX} , then

$$|A_m| = \frac{2}{N} |X(mN_p + 1)| \quad 0 \leq m \leq h - 1$$

and $X(k) = 0$ for all k not defined above

where $\langle X(k) \rangle_{k=0}^{N-1}$ is the DFT of the sequence $\langle x(kT_s) \rangle_{k=0}^{N-1}$

$$f = 1/T, \quad f_{MAX} = \frac{f}{2} \cdot \left\lceil \frac{N}{N_p} \right\rceil, \text{ and } h = \text{Int} \left(\left\lceil \frac{N}{2} - 1 \right\rceil \frac{1}{N_p} \right)$$

Question: Why are we limiting our inputs to periodic signals?

We are looking for metrics for characterizing the linearity of a data converter

FFT Examples

Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required integral number of periods and sampling rate to exceed the Nyquist rate

1. The sampling window be an integral number of periods

2.
$$N > \frac{2 f_{\max}}{f_{\text{SIG}}} N_P$$

from:

$$\left\{ \begin{array}{l} f_{\text{SAMP}} > 2f_{\max} \\ NT_{\text{SAMP}} = N_P T_{\text{SIG}} \\ Nf_{\text{SIG}} = N_P f_{\text{SAMP}} \\ f_{\text{MAX}} \leq \frac{f_{\text{SIG}}}{2} \cdot \left\lceil \frac{N}{N_P} \right\rceil \end{array} \right.$$

Considerations for Spectral Characterization

- Tool Validation
- FFT Length
- Importance of Satisfying Hypothesis
- Windowing

Considerations for Spectral Characterization



- Tool Validation (MATLAB)
- FFT Length
- Importance of Satisfying Hypothesis
- Windowing

FFT Examples

Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required



1. The sampling window must be an integral number of periods
2.
$$N > \frac{2 f_{\max}}{f_{\text{SIG}}} N_P$$

Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_p=11$ $N=512$

Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

$$f_{\text{MAX-ACT}}=100\text{Hz}$$

Consider $N_p=11$ $N=512$

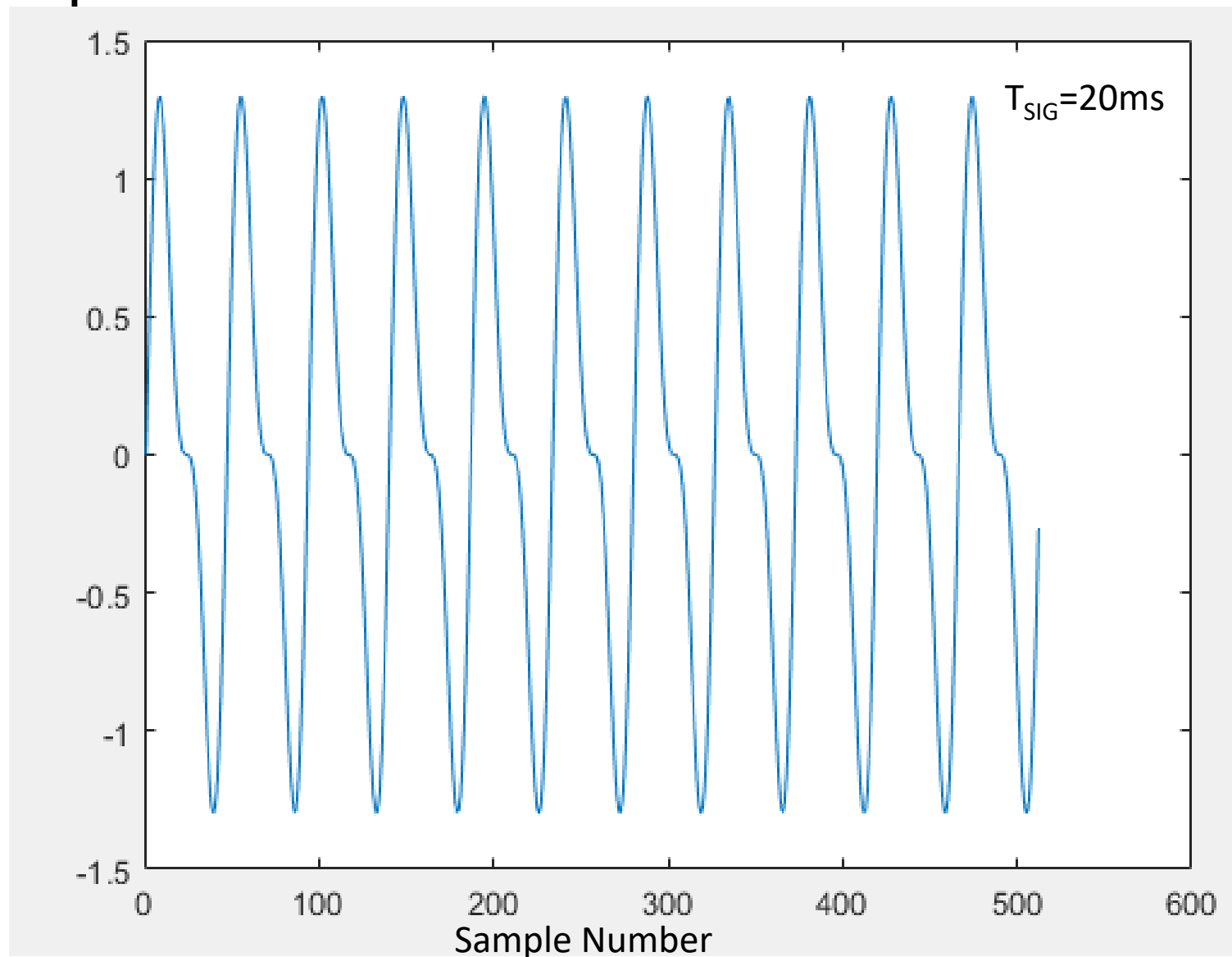
$$f_{\text{MAX}} = \frac{f_{\text{SIG}}}{2} \cdot \left[\frac{N}{N_p} \right] = \frac{50}{2} \cdot \frac{512}{11} = 1.164 \text{ KHz} \quad f_{\text{MAX-ACT}} \ll f_{\text{MAX}}$$

$$f_{\text{SAMPLE}} = \frac{1}{T_{\text{SAMPLE}}} = \frac{1}{\left(\frac{N_p \cdot T_{\text{SIG}}}{N} \right)} = \left[\frac{N}{N_p} \right] f_{\text{SIG}} = 2f_{\text{MAX}} = 2.327... \text{ KHz}$$

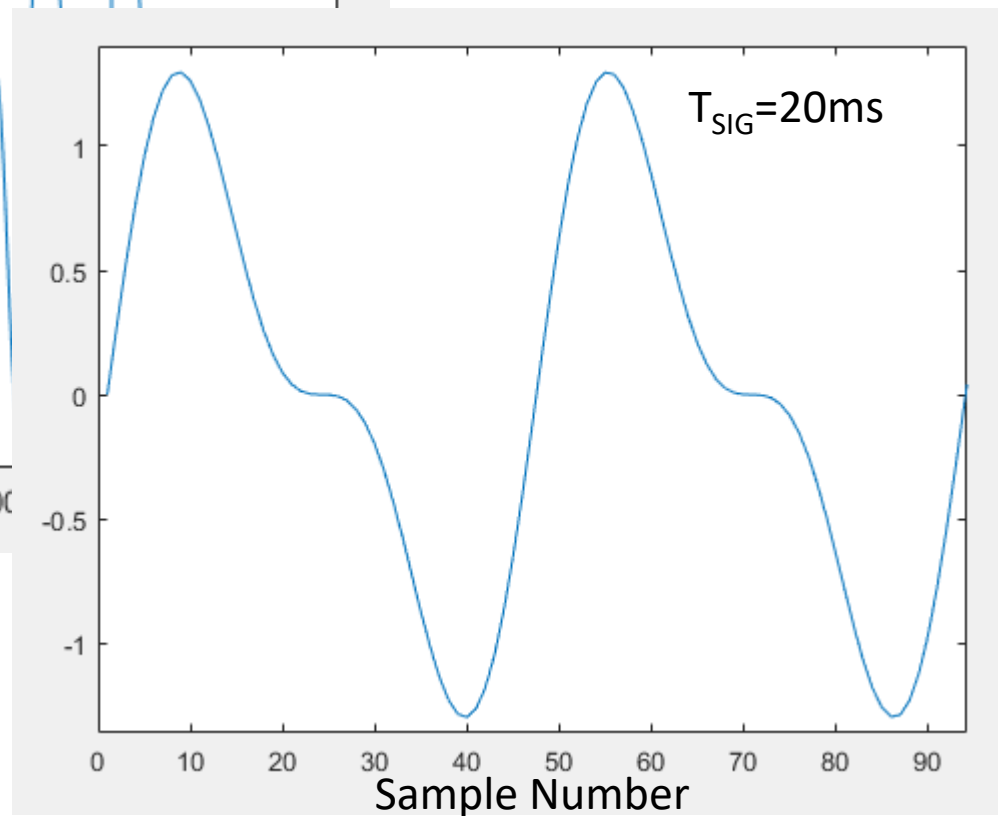
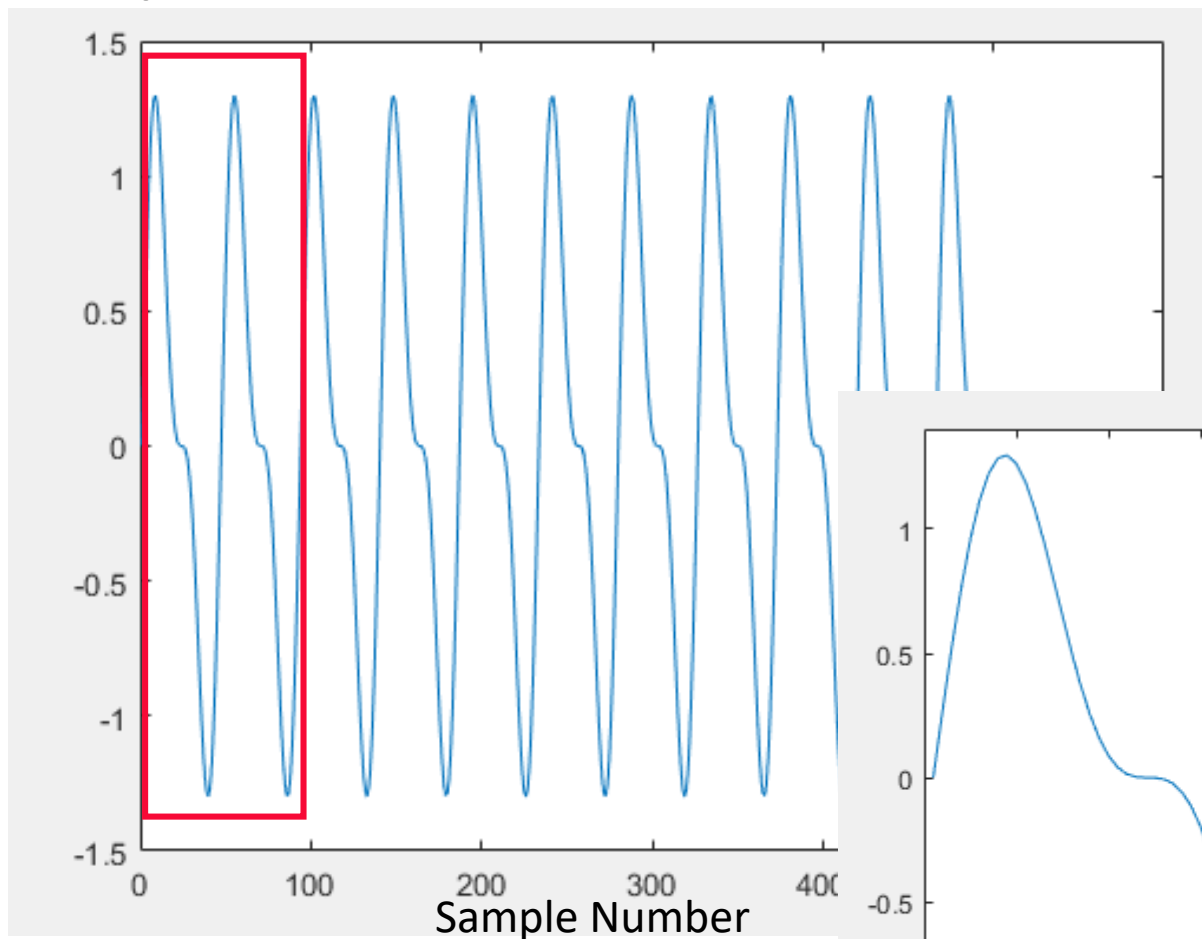
Recall $20\log_{10}(1.0)=0.000000$

Recall $20\log_{10}(0.5)=-6.0205999$

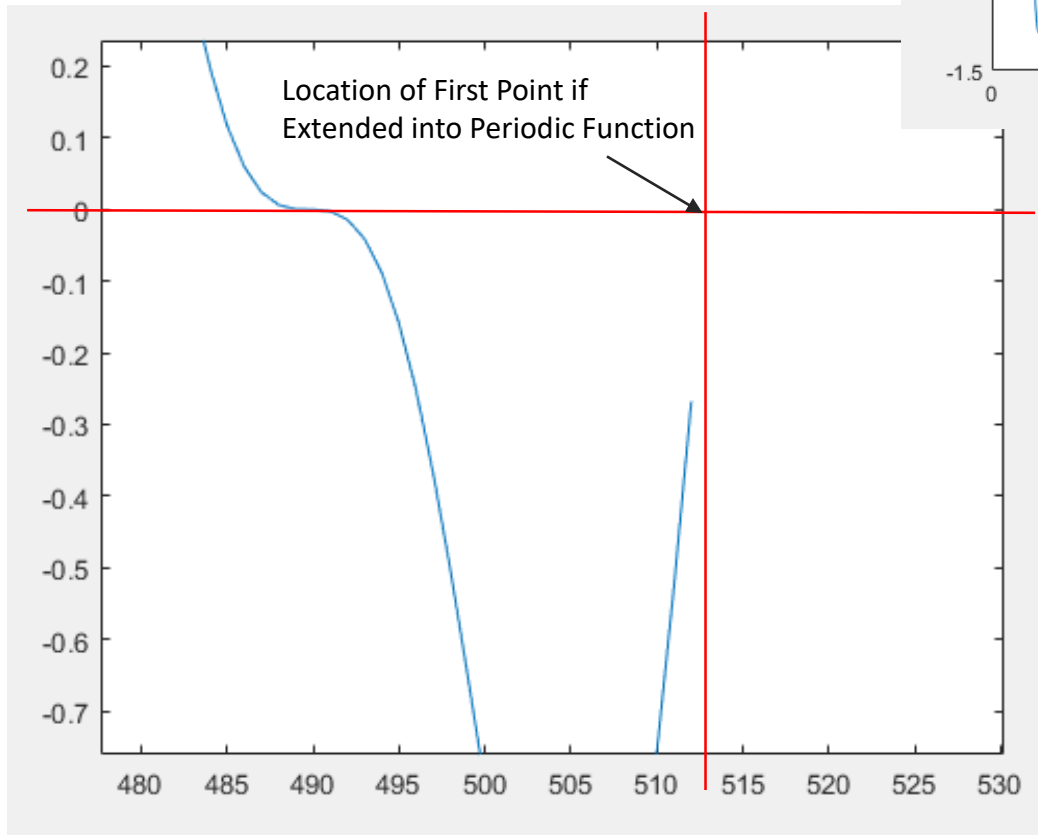
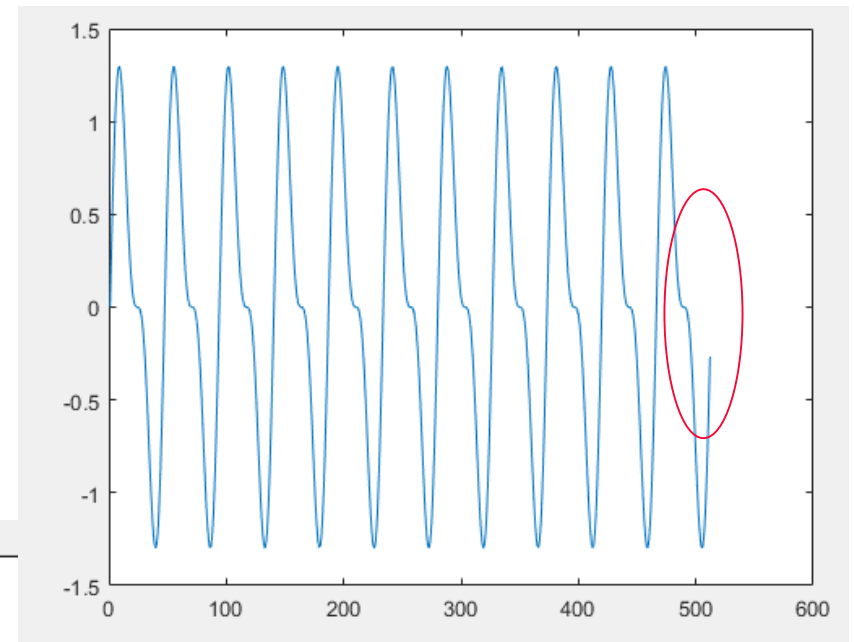
Input Waveform



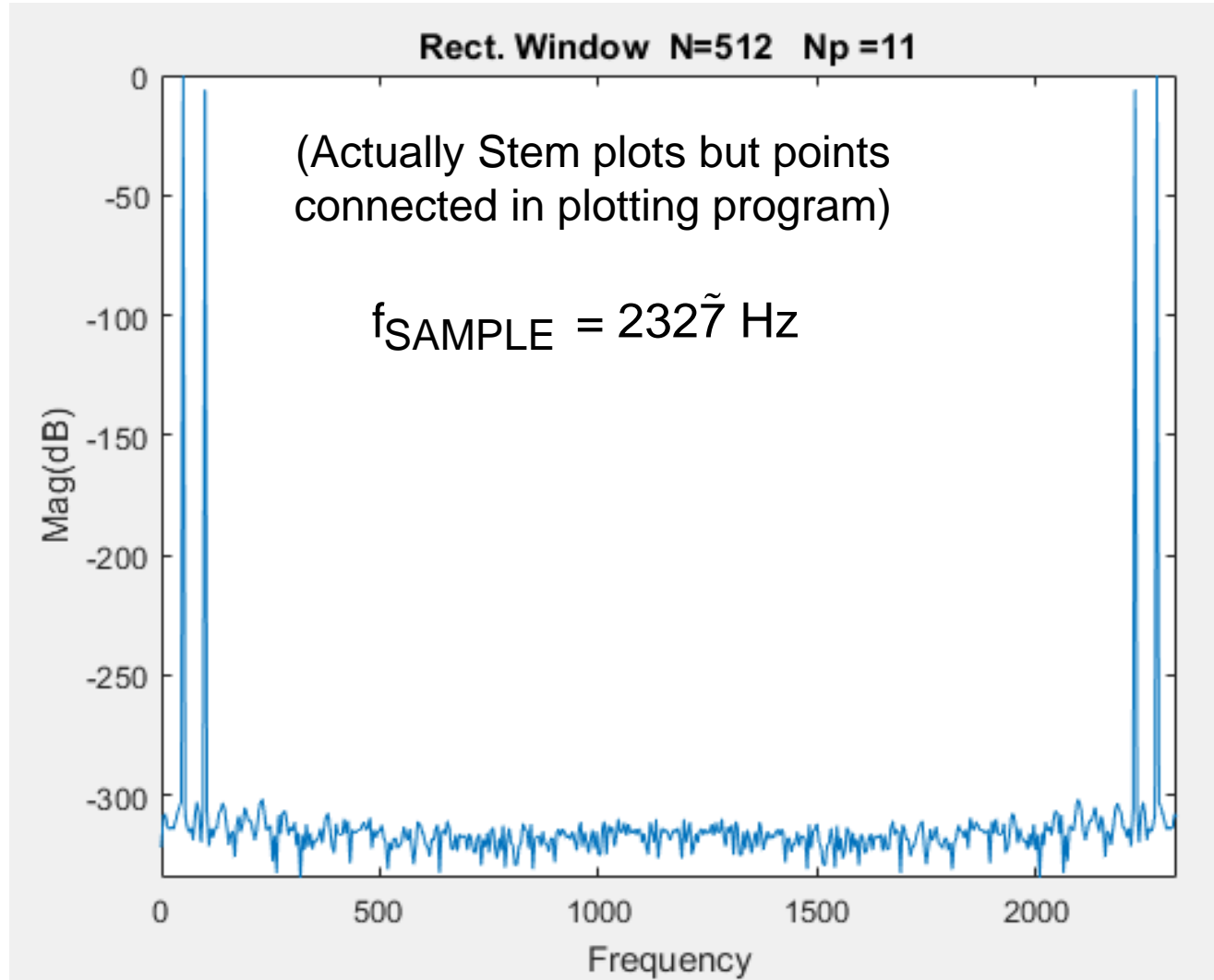
Input Waveform



Input Waveform

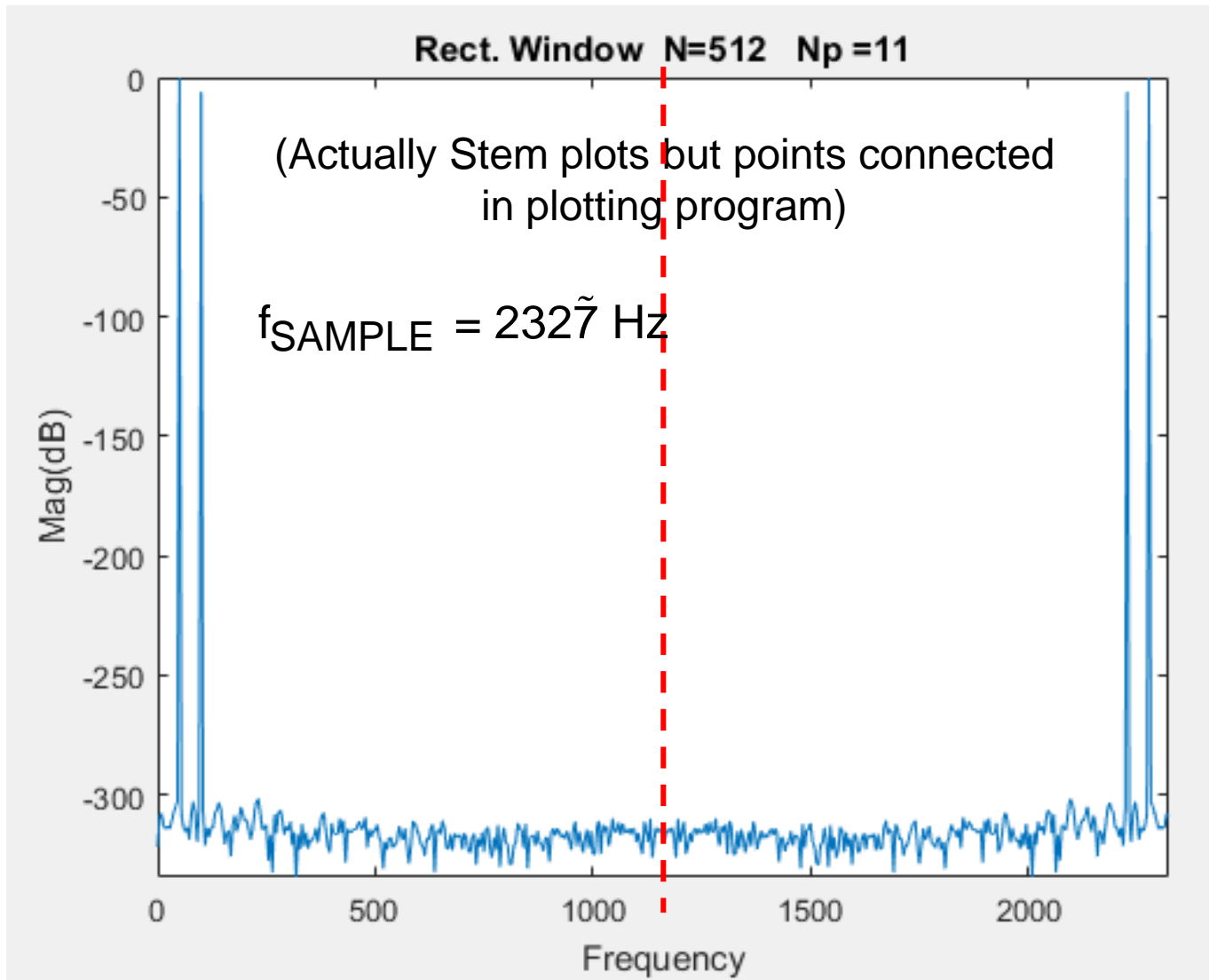


Spectral Response from MATLAB (expressed in dB)



(Horizontal axis is the “Index” axis (k) but converted to frequency) $f_{\text{AXIS}} = f_{\text{SIGNAL}} \frac{k-1}{N_P}$

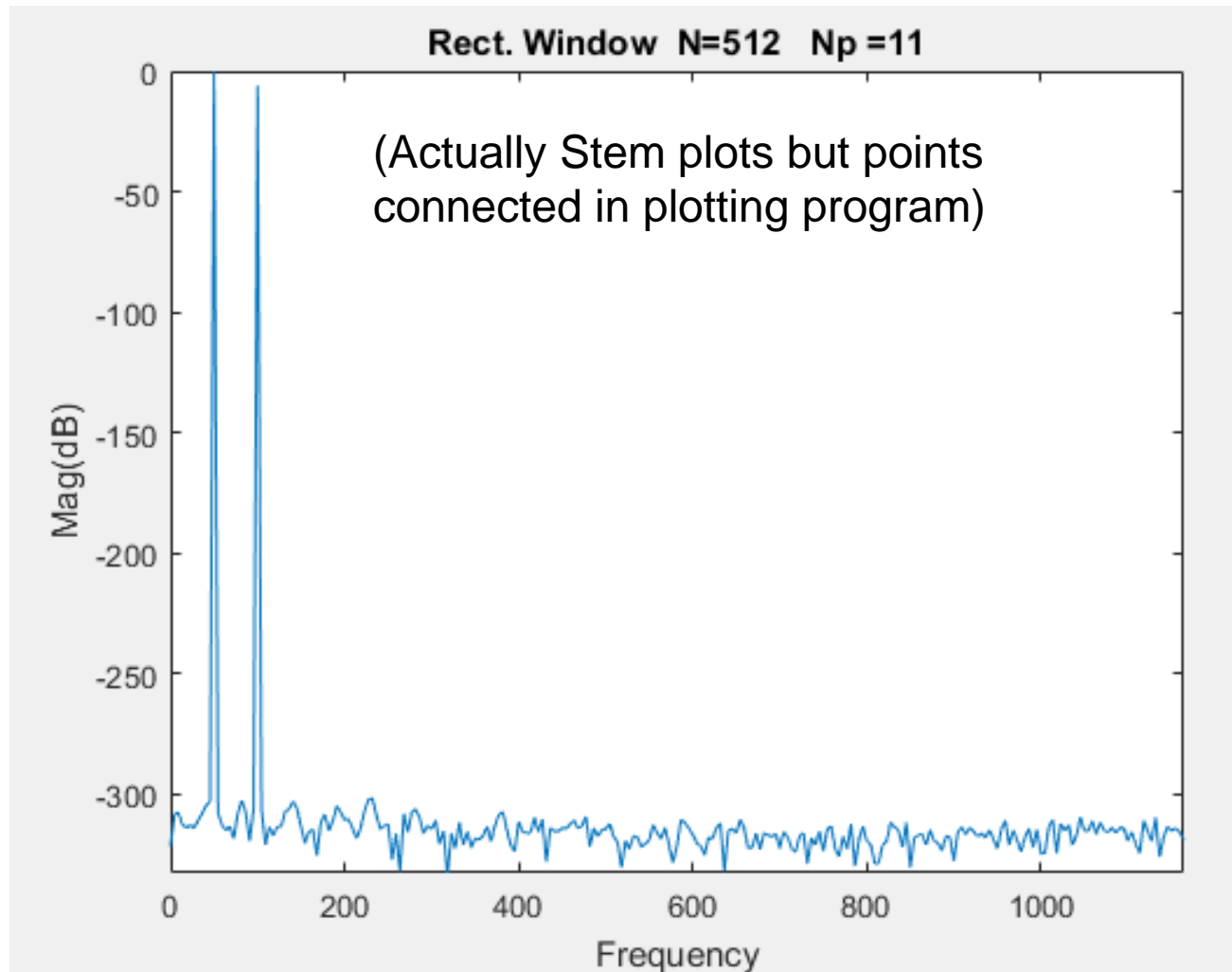
Spectral Response (expressed in dB)



Note Magnitude is Symmetric wrt $f_{\text{SAMPLE}}/2$

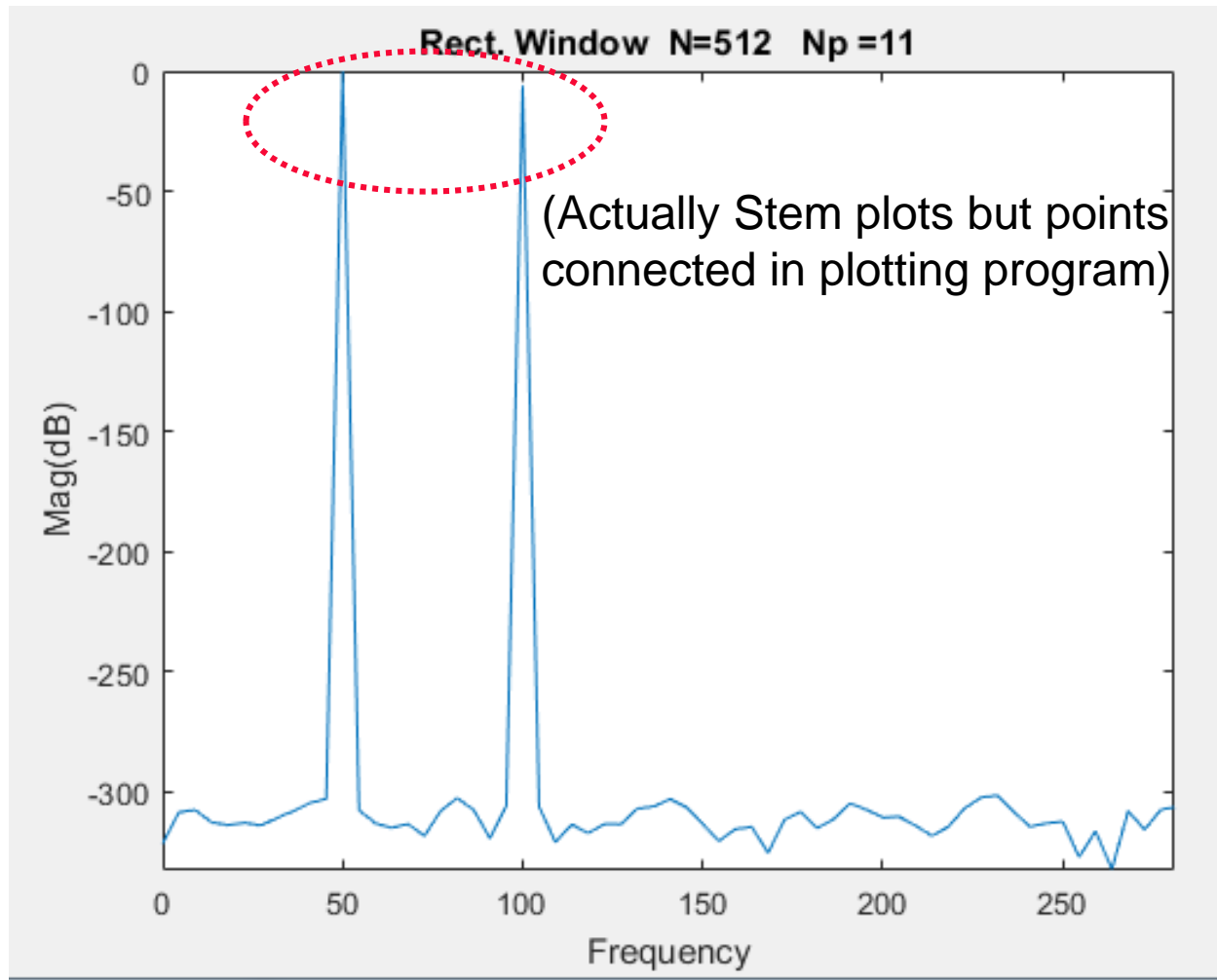
$$f_{\text{AXIS}} = f_{\text{SIGNAL}} \frac{k-1}{N_P}$$

Spectral Response (expressed in dB)



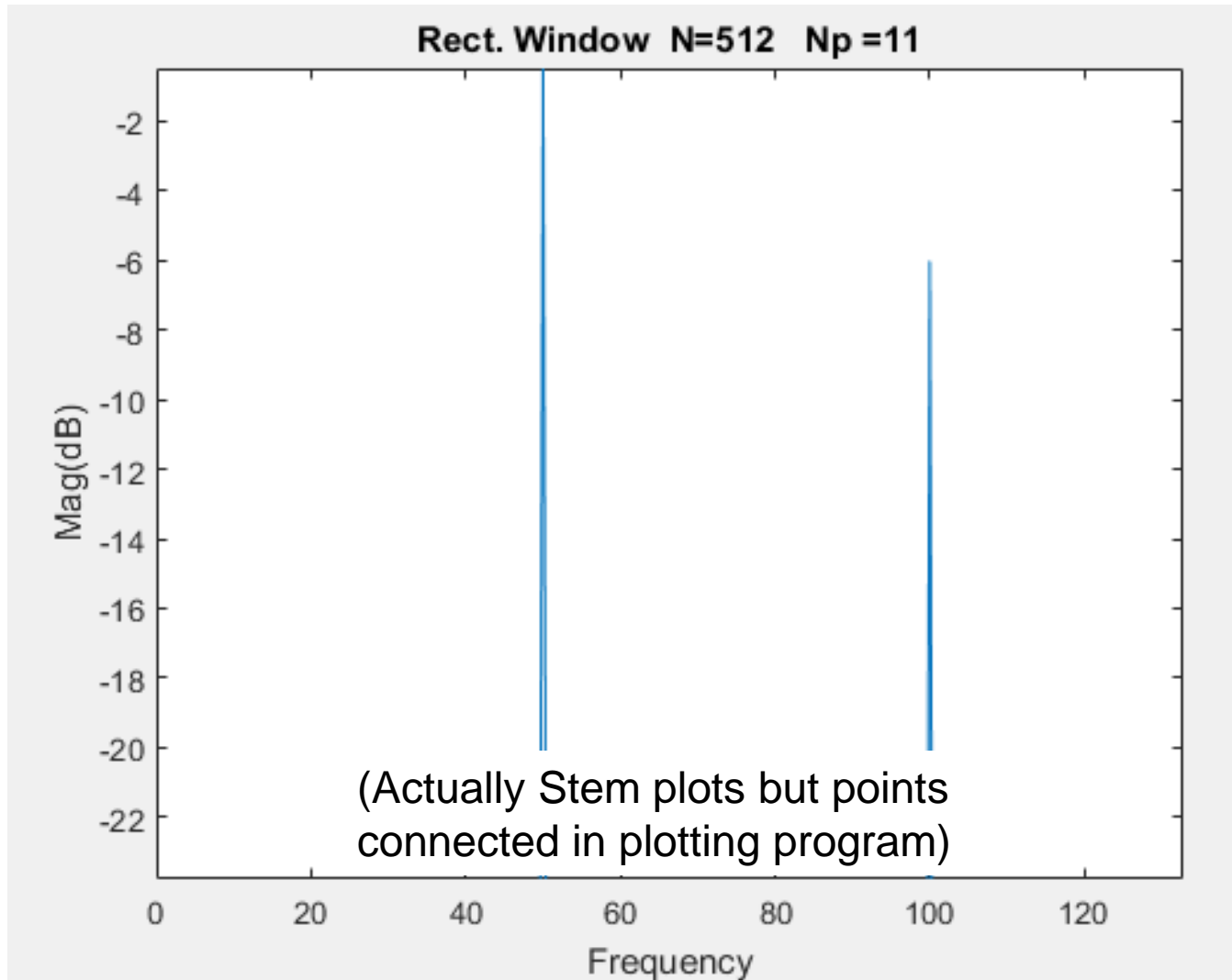
$$f_{\text{AXIS}} = f_{\text{SIGNAL}} \frac{k-1}{N_P}$$

Spectral Response



$$f_{AXIS} = f_{SIGNAL} \frac{k-1}{N_P}$$

Spectral Response



$$f_{\text{AXIS}} = f_{\text{SIGNAL}} \frac{k-1}{N_P}$$

Fundamental will appear at position $1+N_p = 12$

Columns 1 through 11

-321.7231 -308.6975 -307.7331 -312.9228 -314.0436 -313.1052 -314.1937 -311.0721 -308.1500 -304.8602 -303.1474

Columns 12 through 22

-0.0000 -307.8133 -313.2869 -315.1953 -313.5456 -318.5818 -308.0412 -302.7892 -307.6748 -319.5537 -306.1232

Columns 23 through 33

-6.0206 -306.3159 -321.1897 -313.7555 -317.4130 -313.5698 -313.6783 -307.1848 -306.3061 -303.1651 -306.5553

Columns 34 through 44

-313.5577 -320.6716 -315.7210 -314.8459 -325.6765 -311.7824 -308.5404 -315.3695 -311.5252 -304.9991 -307.6242

Columns 45 through 55

-310.8716 -310.5432 -314.2436 -318.5144 -314.7917 -307.0888 -302.5236 -301.7044 -308.4356 -314.7118 -313.2992

Observe system noise floor due to both spectral limitations of signal generator and numerical limitations in FFT are below -300db

Second Harmonic at $1+2Np = 23$

Columns 1 through 11

-321.7231 -308.6975 -307.7331 -312.9228 -314.0436 -313.1052 -314.1937 -311.0721 -308.1500 -304.8602 -303.1474

Columns 12 through 22

-0.0000 -307.8133 -313.2869 -315.1953 -313.5456 -318.5818 -308.0412 -302.7892 -307.6748 -319.5537 -306.1232

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-313.5577 -320.6716 -315.7210 -314.8459 -325.6765 -311.7824 -308.5404 -315.3695 -311.5252 -304.9991 -307.6242

Columns 45 through 55

-310.8716 -310.5432 -314.2436 -318.5144 -314.7917 -307.0888 -302.5236 -301.7044 -308.4356 -314.7118 -313.2992

Recall $20\log_{10}(0.5)=-6.0205999$

Third Harmonic at $1+3Np = 34$

Columns 1 through 11

-321.7231 -308.6975 -307.7331 -312.9228 -314.0436 -313.1052 -314.1937 -311.0721 -308.1500 -304.8602 -303.1474

Columns 12 through 22

-0.0000 -307.8133 -313.2869 -315.1953 -313.5456 -318.5818 -308.0412 -302.7892 -307.6748 -319.5537 -306.1232

Columns 23 through 33

-6.0206 -306.3159 -321.1897 -313.7555 -317.4130 -313.5698 -313.6783 -307.1848 -306.3061 -303.1651 -306.5553

Columns 34 through 44

-313.5577 -320.6716 -315.7210 -314.8459 -325.6765 -311.7824 -308.5404 -315.3695 -311.5252 -304.9991 -307.6242

Columns 45 through 55

-310.8716 -310.5432 -314.2436 -318.5144 -314.7917 -307.0888 -302.5236 -301.7044 -308.4356 -314.7118 -313.2992

Example - Increasing N_p

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_p=31$ $N=512$

Example – Increasing N_p

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

$$f_{\text{MAX-ACT}}=100\text{Hz}$$

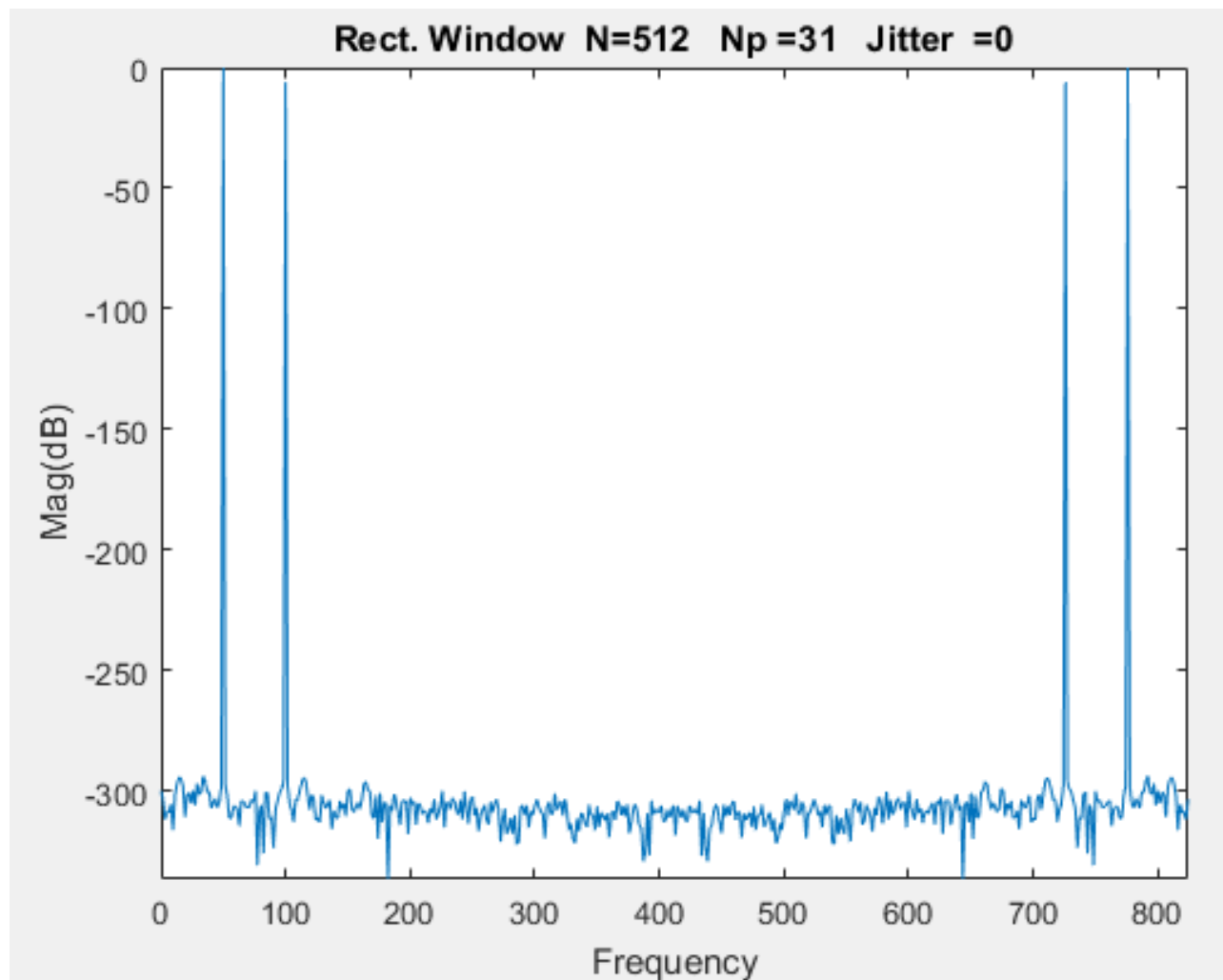
Consider $N_p=31$ $N=512$

$$f_{\text{MAX}} = \frac{f_{\text{SIG}}}{2} \cdot \left\lceil \frac{N}{N_p} \right\rceil = \frac{50}{2} \cdot \frac{512}{31} = 412.9 \text{ Hz} \quad f_{\text{MAX-ACT}} \ll f_{\text{MAX}}$$

$$f_{\text{SAMPLE}} = \frac{1}{T_{\text{SAMPLE}}} = \frac{1}{\left(\frac{N_p \cdot T_{\text{SIG}}}{N} \right)} = \left\lceil \frac{N}{N_p} \right\rceil f_{\text{SIG}} = 2f_{\text{MAX}} = 825.8 \text{ Hz}$$

Recall $20\log_{10}(1.0)=0.000000$

Recall $20\log_{10}(0.5)=-6.0205999$



Fundamental will appear at position $1+Np = 32$

Columns 1 through 11

-299.6472 -303.2647 -311.4997 -308.3073 -308.8025 -305.2792 -315.9958 -301.4260 -296.7862 -294.0674 -295.0438

Columns 12 through 22

-298.0298 -310.5850 -301.1461 -300.4606 -304.9693 -299.6351 -305.7190 -296.6982 -301.2474 -299.9385 -293.5499

Columns 23 through 33

-295.9396 -300.3529 -299.9751 -306.9248 -304.7769 -302.8556 -306.7452 -304.6290 -297.8691 -0.0000 -297.2993

Columns 34 through 44

-302.0105 -311.1106 -311.1480 -306.3087 -306.2084 -307.5956 -314.2196 -303.7547 -304.1900 -305.9837 -306.5840

Columns 45 through 55

-306.2397 -303.5875 -304.1769 -299.7907 -330.5678 -307.8635 -305.5380 -325.6389 -300.4851 -300.7637 -311.4764

Columns 56 through 66

-310.5216 -323.3310 -307.7034 -303.6633 -300.5140 -298.8582 -296.0678 -6.0206 -297.4520 -304.9769 -307.6024

Columns 67 through 77

-312.4863 -303.4440 -303.5171 -299.4001 -296.2720 -294.2365 -295.5554 -301.2603 -307.5154 -301.2971 -302.8147

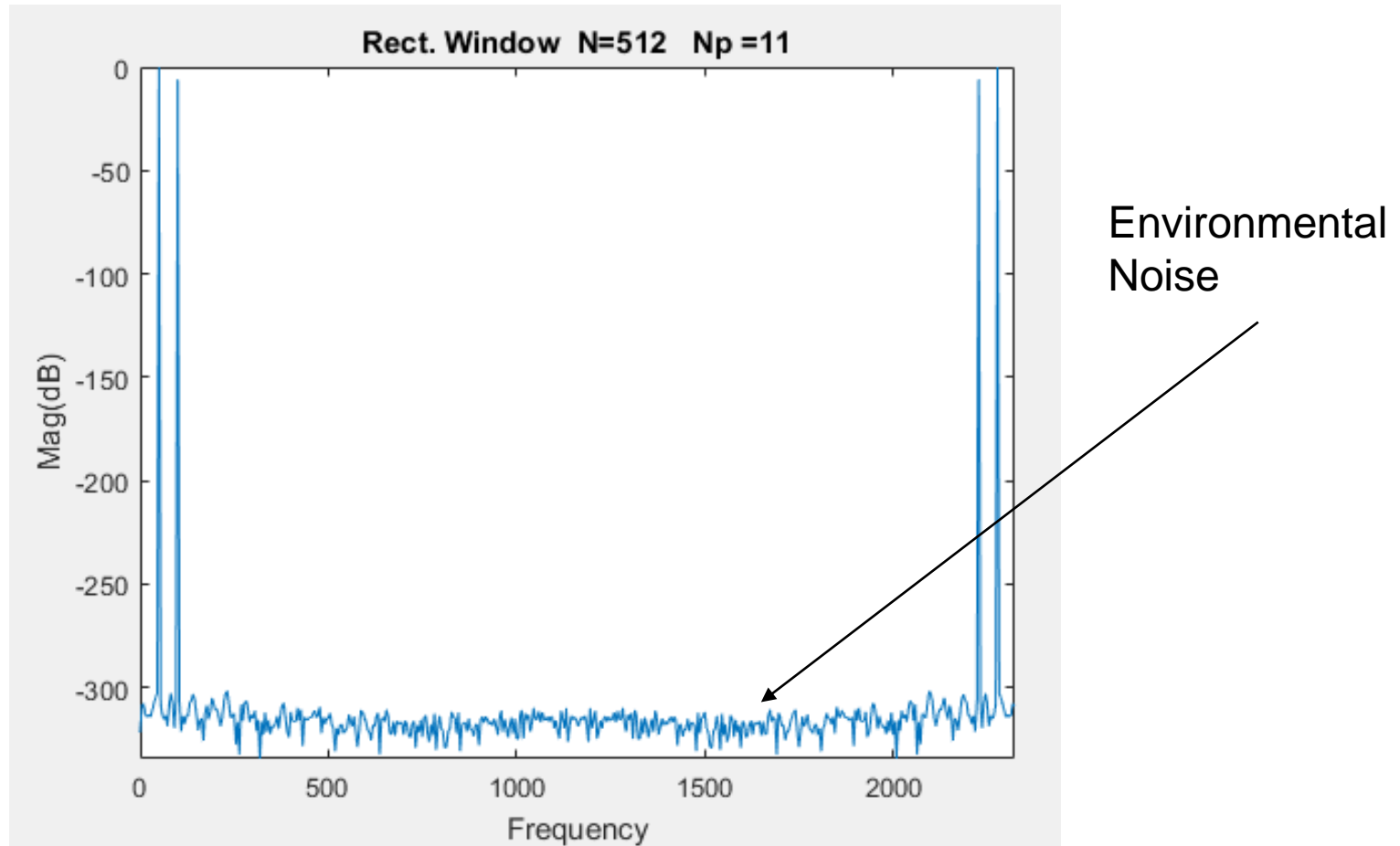
Columns 78 through 88

-311.7453 -310.8834 -312.7745 -301.5065 -304.5661 -306.9176 -305.4165 -303.5872 -315.6237 -308.0496 -310.6269

Columns 89 through 99

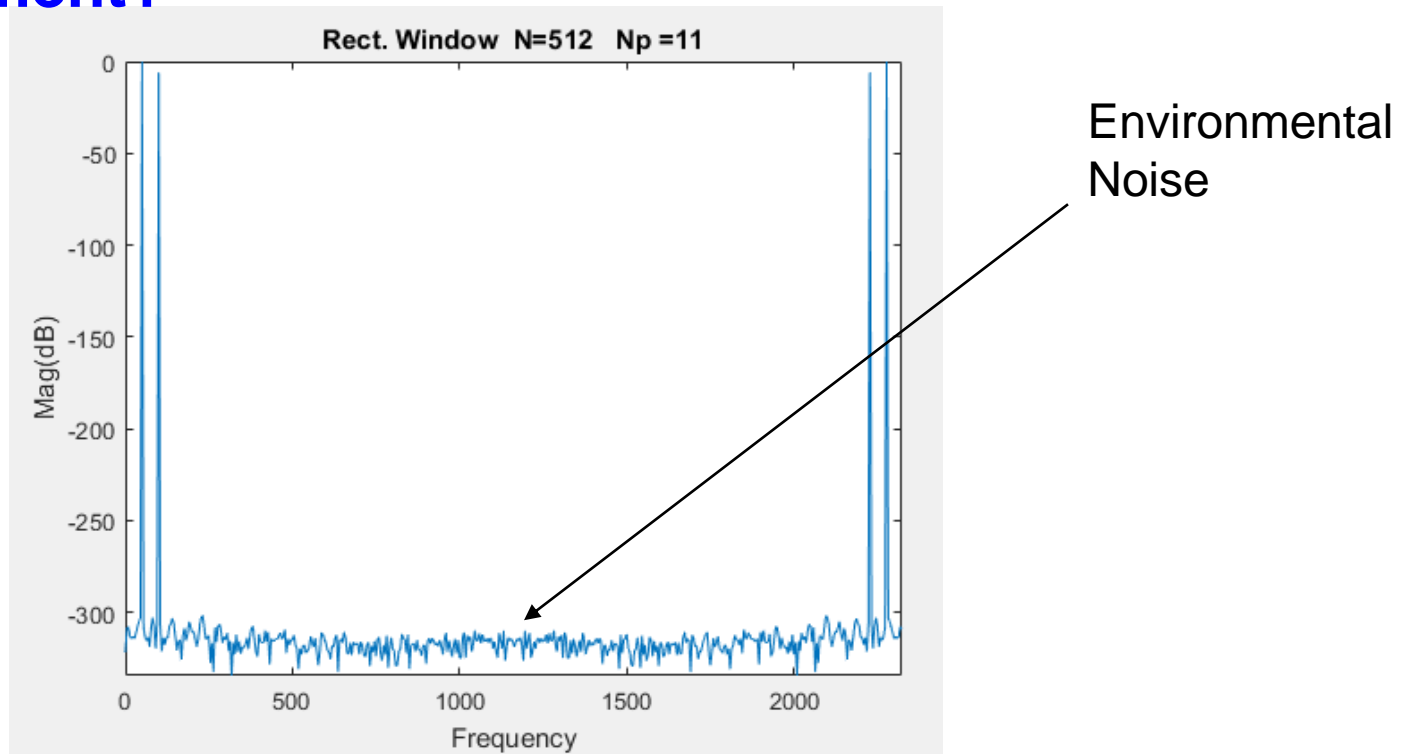
-311.1097 -303.8470 -307.2403 -304.9455 -310.2498 -299.7722 -298.9617 -307.3191 -308.4678 -306.2355 -304.7098

Question: How much noise is in the computational environment?



Is this due to quantization in the computational environment?
to numerical rounding in the FFT calculation?
to errors in calculating $\sin(x)$?

Question: How much noise is in the computational environment?



Observation: This noise is nearly uniformly distributed
The level of this noise at each component is around -310dB

Question: How much noise is in the computational environment?

Assume $A_k = -310$ dB for $0 \leq k \leq N$

$$A_{\text{kdB}} = 20 \log_{10} A_k \quad \longrightarrow \quad A_k = 10^{\frac{A_{\text{kdB}}}{20}}$$

$$A_k \cong 10^{\frac{-310}{20}} = 10^{-15.5} \stackrel{\text{defn}}{=} \bar{A}$$

$$V_{\text{Noise,RMS}} \cong \sqrt{\sum_{k=1}^{N-1} \left(\frac{A_k}{\sqrt{2}} \right)^2} \stackrel{\substack{A_k = \bar{A} \\ N \text{ large}}}{=} \bar{A} \sqrt{\frac{N}{2}}$$

$$V_{\text{Noise,RMS}} \cong \bar{A} \sqrt{\frac{N}{2}} = 10^{-15.5} \sqrt{\frac{512}{2}} = 5.1 \cdot 10^{-15} \cong 5 \text{ fV}$$

Note: **This computational environment has a very low total computational noise and does not become significant until the 46-bit resolution level is reached !!**

Considerations for Spectral Characterization

- Tool Validation



- FFT Length

- Importance of Satisfying Hypothesis

- Windowing

Example – Increasing N

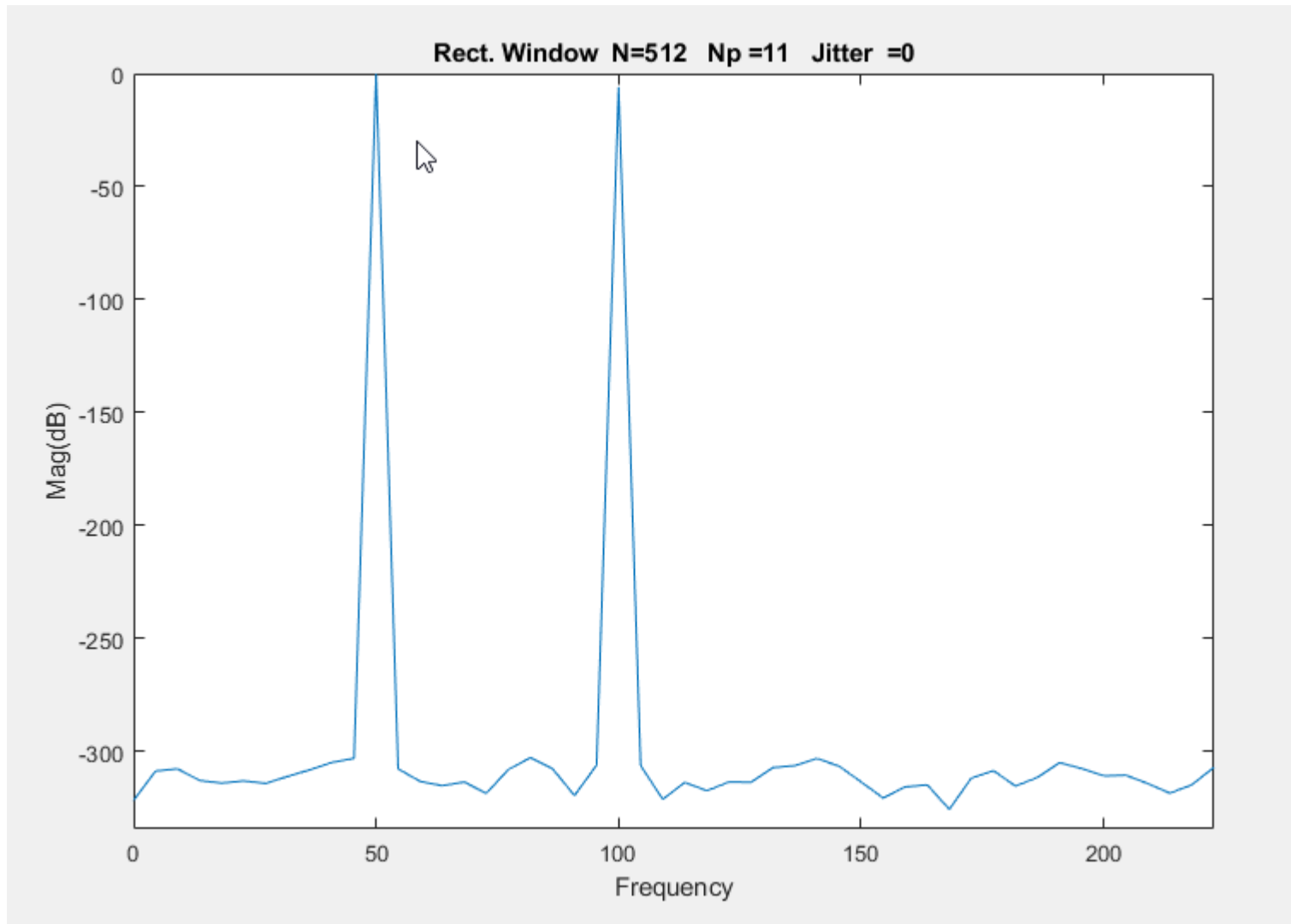
WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Recall $N_p=11$ $N=512$

Spectral Response



DFT Horizontal Axis Converter to Frequency : $f_{\text{AXIS}} = f_{\text{SIGNAL}} \frac{n-1}{N_p}$

Fundamental will appear at position $1+N_p = 12$
Second harmonic will appear at position $1+2N_p=23$

Columns 1 through 5

-321.7231 -308.6975 -307.7331 -312.9228 -314.0436

Columns 6 through 10

-313.1052 -314.1937 -311.0721 -308.1500 -304.8602

Columns 11 through 15

-303.1474 -0.0000 -307.8133 -313.2869 -315.1953

Columns 16 through 20

-313.5456 -318.5818 -308.0412 -302.7892 -307.6748

Columns 21 through 25

-319.5537 -306.1232 -6.0206 -306.3159 -321.1897

Recall system noise floor due to both spectral limitations of signal generator and numerical limitations in FFT are below -300db

Example – Increasing N

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

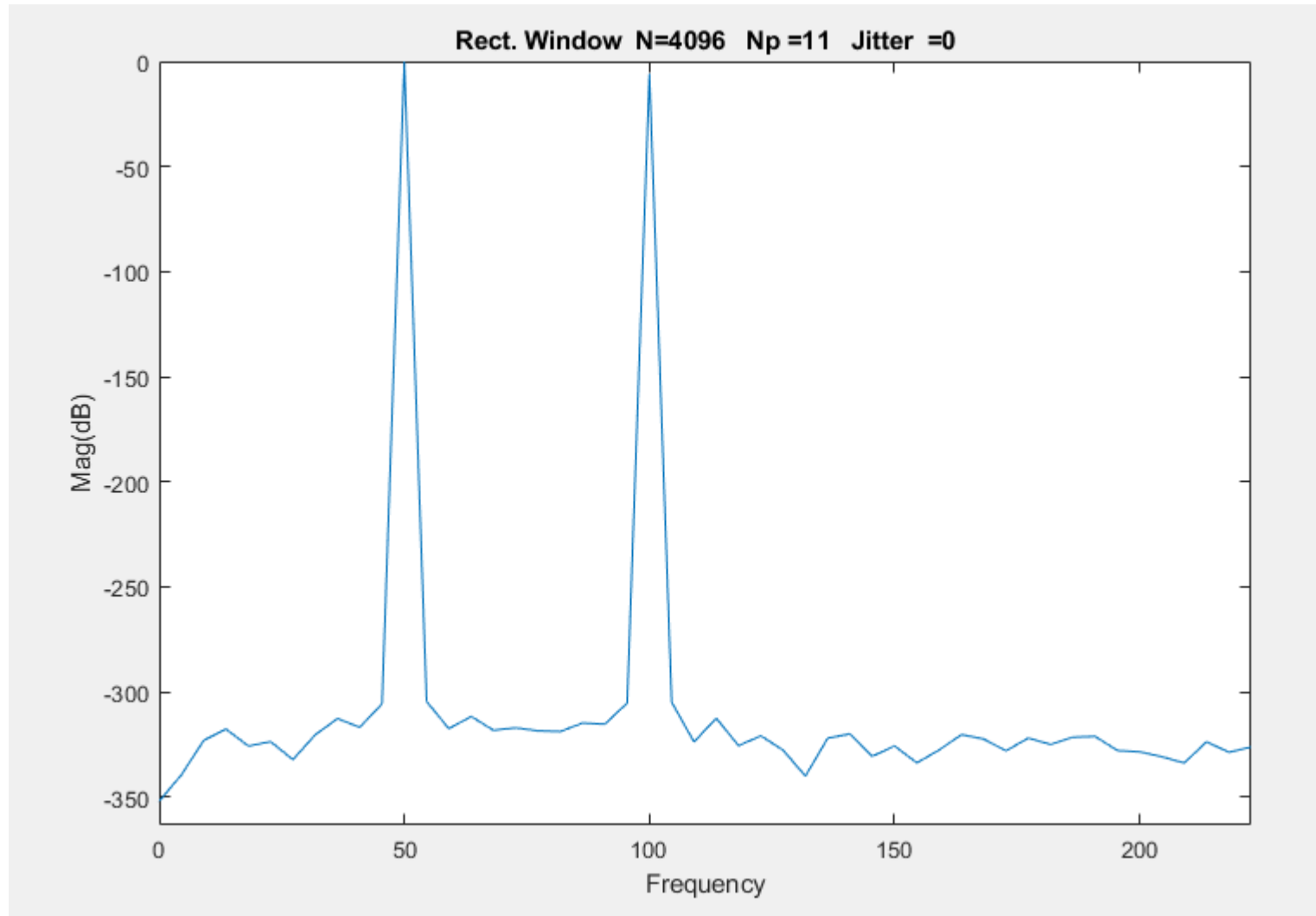
$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

- Increase length from 512 to 4096

Consider $N_p=11$ $N=4096$

Spectral Response



Fundamental will appear at position $1+Np = 12$

For $Np=11$

Second harmonic will appear at position $1+2Np = 23$

Columns 1 through 5

-351.9112 -339.3553 -322.9718 -317.6116 -325.7202

Columns 6 through 11

-323.6033 -332.1468 -320.3687 -312.6237 -316.7584

Columns 12 through 15

-305.5856 0 -304.4897 -317.4074 -311.6053

Columns 16 through 20

-318.1774 -317.0160 -318.4626 -318.7780 -314.7295

Columns 21 through 25

-315.2757 -305.2332 -6.0206 -304.8544 -323.7100

Third harmonic will appear at position $1+3N_p = 34$

For $N_p=11$

Columns 26 through 30

-312.4569 -325.5267 -320.8299 -327.6482 -340.0002

Columns 31 through 35

-321.9740 -319.8910 -330.5761 -325.5995 -333.6951

Example - Increasing N_p

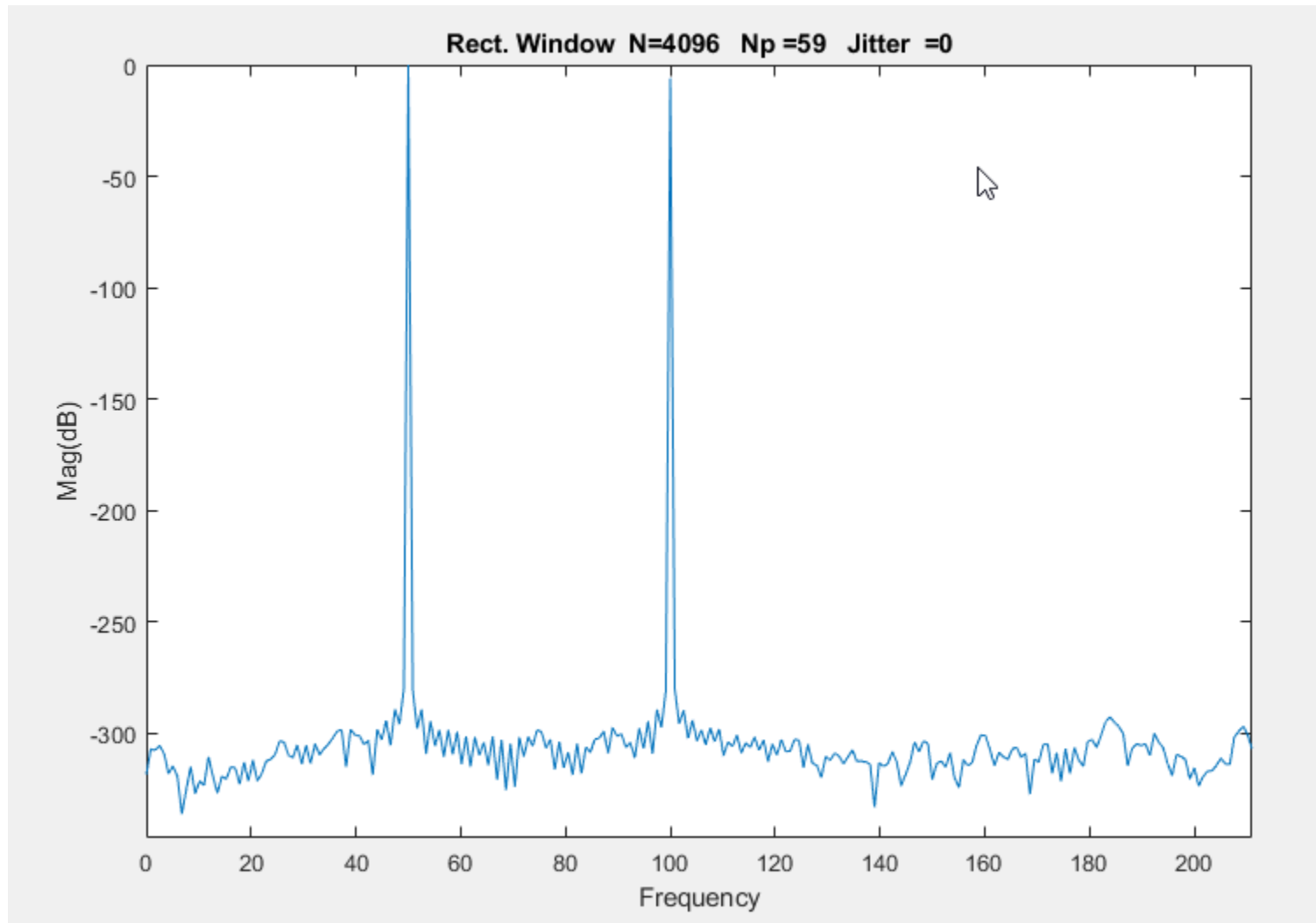
WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_p=59$ $N=4096$

Spectral Response



Fundamental will appear at position $1+N_p = 60$

$N_p=59$ $N=4096$

Columns 1 through 13

-318.5027 -307.1222 -307.5852 -305.4679 -309.0657 -318.1047 -314.6599 -319.2843 -336.2985 -325.1317
-315.0935 -327.1326 -321.3676

Columns 14 through 26

-323.4308 -310.6280 -319.2470 -326.8257 -319.4502 -320.5664 -315.1544 -315.5020 -322.8177 -313.0289
-321.4672 -311.9602 -321.4265

Columns 27 through 39

-318.1381 -312.5542 -311.6208 -309.5333 -303.4393 -304.0017 -309.8193 -311.0570 -305.3422 -313.8205
-305.4006 -313.4782 -304.6525

Columns 40 through 52

-309.6946 -306.9554 -304.8106 -302.1759 -299.0381 -298.4703 -315.1285 -298.2425 -300.8413 -301.0957
-305.0282 -303.3066 -318.7951

Columns 53 through 65

-298.2648 -302.9334 -294.3147 -305.4528 -289.2434 -295.8942 -280.1798 -0.0000 -280.5964 -297.8514 -
289.3267 -309.3347 -294.5964

Second Harmonic will appear at position $1+2N_p = 119$

$N_p=59$ $N=4096$

Columns 92 through 104

-302.8127 -316.2949 -303.7744 -315.5271 -308.4771 -318.7071 -304.6313 -318.0416
-306.1670 -308.5714 -302.7302 -302.2368 -299.1027

Columns 105 through 117

-308.9464 -297.3257 -301.3703 -300.2894 -306.4180 -304.1136 -311.0688 -297.7426
-306.8213 -294.6257 -309.2568 -289.3121 -297.4441

Columns 118 through 130

-280.6899 **-6.0206** -280.0699 -295.8322 -289.6408 -302.2283 -294.1945 -303.4744 -
298.5343 -305.2095 -297.5745 -303.8141 -298.1884

Columns 131 through 143

-310.0974 -303.9414 -305.9561 -300.8293 -308.8257 -304.3097 -306.3100 -301.7031
-307.6105 -303.0237 -312.6641 -304.8315 -309.8784

Third Harmonic will appear at position $1+3N_p = 178$

$N_p=59$ $N=4096$

Columns 157 through 169

-310.2143 -313.7396 -310.6594 -307.4932 -312.5744 -312.4317 -312.9279 -313.8457
-333.2123 -313.2139 -314.8757 -313.6770 -308.2654

Columns 170 through 182

-312.9108 -323.6234 -318.4068 -313.1440 -303.9230 -308.4125 -303.5257 -304.4918
-320.8338 -313.5467 -312.3853 -315.2628 -308.7278

Columns 183 through 195

-319.9543 -324.3752 -311.8755 -314.5450 -313.2239 -305.2555 -300.7224 -301.1777
-307.2956 -314.6381 -308.5318 -310.5178 -311.8403

Columns 196 through 208

-307.0945 -306.2901 -310.7842 -309.0464 -327.4021 -311.5712 -312.9993 -305.0627
-304.9283 -317.9667 -308.8503 -321.4805 -306.9463

Fundamental will appear at position $1+N_p = 60$

$N_p=59$ $N=4096$

Has the environmental noise floor increased?

Columns 1 through 13

-318.5027 -307.1222 -307.5852 -305.4679 -309.0657 -318.1047 -314.6599 -319.2843 -336.2985 -325.1317
-315.0935 -327.1326 -321.3676

Columns 14 through 26

-323.4308 -310.6280 -319.2470 -326.8257 -319.4502 -320.5664 -315.1544 -315.5020 -322.8177 -313.0289
-321.4672 -311.9602 -321.4265

Columns 27 through 39

-318.1381 -312.5542 -311.6208 -309.5333 -303.4393 -304.0017 -309.8193 -311.0570 -305.3422 -313.8205
-305.4006 -313.4782 -304.6525

Columns 40 through 52

-309.6946 -306.9554 -304.8106 -302.1759 -299.0381 -298.4703 -315.1285 -298.2425 -300.8413 -301.0957
-305.0282 -303.3066 -318.7951

Columns 53 through 65

-298.2648 -302.9334 -294.3147 -305.4528 -289.2434 -295.8942 -280.1798 -0.0000 280.5964 -297.8514 -
289.3267 -309.3347 -294.5964

Second Harmonic will appear at position $1+2N_p = 119$ $N_p=59$ $N=4096$

Has the environmental noise floor increased?

Columns 92 through 104

-302.8127 -316.2949 -303.7744 -315.5271 -308.4771 -318.7071 -304.6313 -318.0416
-306.1670 -308.5714 -302.7302 -302.2368 -299.1027

Columns 105 through 117

-308.9464 -297.3257 -301.3703 -300.2894 -306.4180 -304.1136 -311.0688 -297.7426
-306.8213 -294.6257 -309.2568 -289.3121 -297.4441

Columns 118 through 130

-280.6899 -6.0206 -280.0699 -295.8322 -289.6408 -302.2283 -294.1945 -303.4744 -
298.5343 -305.2095 -297.5745 -303.8141 -298.1884

Columns 131 through 143

-310.0974 -303.9414 -305.9561 -300.8293 -308.8257 -304.3097 -306.3100 -301.7031
-307.6105 -303.0237 -312.6641 -304.8315 -309.8784

Has the environmental noise floor increased?

Example - Increasing N_p

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_p=59$ $N=65536$

Has the environmental noise floor increased?

Fundamental will appear at position $1+N_p=60$

$N_p=59$ $N=65536$

Columns 1 through 13

-334.8087 -306.4282 -305.3385 -306.0729 -310.4797 -319.1995 -319.2015 -326.9957 -323.5461
-320.9973 -322.3010 -330.1711 -341.5590

Columns 14 through 26

-335.6793 -324.5727 -331.9776 -328.7216 -328.2566 -322.5912 -323.1561 -327.1231 -324.6136
-327.6223 -319.4305 -319.3088 -320.0340

Columns 27 through 39

-333.4851 -317.9818 -317.6947 -310.0325 -305.2753 -304.6863 -313.1882 -316.6812 -308.8293
-320.0442 -315.7717 -321.1714 -314.0171

Columns 40 through 52

-311.9394 -313.5694 -312.8475 -313.0765 -311.4848 -309.8760 -311.2895 -317.5812 -311.1151
-305.7421 -310.4045 -307.3691 -308.2844

Columns 53 through 65

-303.0035 -311.0412 -301.9337 -305.0463 -295.1819 -297.5368 -286.7967 0 -287.9392
-297.7360 -295.6029 -303.5877 -302.3842

Has the environmental noise floor increased?

Second Harmonic will appear at position $1+2N_p = 119$ $N_p=59$ $N=65536$

Columns 92 through 104

-306.5163 -315.4662 -307.4725 -316.8662 -318.3799 -350.5195 -314.7483 -
314.9668 -312.8332 -316.2735 -314.6227 -314.1683 -309.7421

Columns 105 through 117

-313.4242 -315.6570 -314.5305 -308.1299 -314.6932 -308.8232 -310.8042 -
302.5240 -312.0286 -301.8591 -305.6318 -296.0575 -298.3169

Columns 118 through 130

-288.0902 -6.0206 -286.7600 -296.7222 -295.8080 -301.0121 -301.8241 -
309.0274 -302.3173 -305.8535 -305.7722 -307.2773 -305.0839

Columns 131 through 143

-310.3001 -312.2301 -309.6370 -306.9460 -308.2486 -311.0093 -310.3350 -
310.4601 -309.4687 -311.5006 -314.0138 -310.7739 -316.4801

Has the environmental noise floor increased?

Example - Increasing N_p

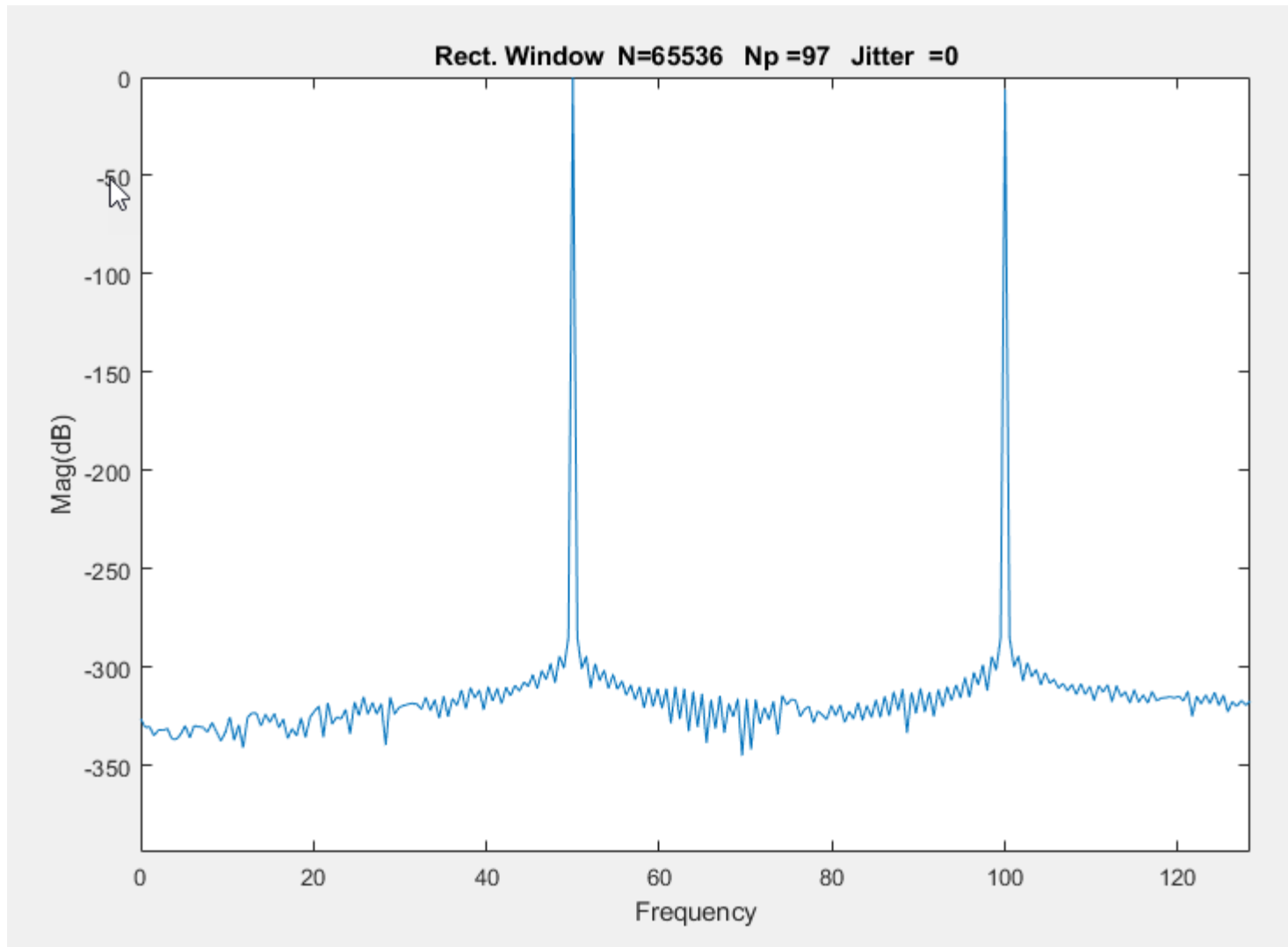
WLOG assume $f_{\text{SIG}}=50\text{Hz}$

$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_p=97$ $N=65536$

Spectral Response



Has the environmental noise floor increased?

Fundamental will appear at position $1+N_p=98$

$N_p=97$ $N=65536$

Columns 79 through 91

-309.9736 -317.2573 -310.8358 -318.6741 -310.3694 -314.6463 -309.3686 -312.1643
-307.6865 -309.9408 -303.7856 -310.9949 -301.7664

Columns 92 through 104

-306.4032 -298.2550 -308.0223 -294.4658 -300.6493 -284.9748 0 -284.9686
-300.8932 -294.4063 -310.6871 -298.2620 -306.9972

Columns 105 through 117

-301.4981 -310.9186 -303.6470 -311.3991 -307.0910 -314.2788 -309.0782 -316.6942
-309.7874 -320.7255 -310.2243 -320.0172 -310.9449

Has the environmental noise floor increased?

Second Harmonic will appear at position $1+2N_p = 195$ $N_p=97$ $N=65536$

Columns 170 through 182

-312.5506 -322.0807 -311.2396 -333.5575 -312.9314 -323.6003 -310.8598 -
322.9989 -312.7526 -325.1455 -311.4532 -320.2566 -310.0410

Columns 183 through 195

-317.0955 -309.0581 -315.8184 -305.1452 -315.2504 -302.7042 -309.1048 -
298.8906 -311.9708 -294.6866 -301.5044 -285.0441 -6.0206

Columns 196 through 208

-284.8891 -299.9611 -294.2706 -307.0905 -297.8530 -304.9661 -301.2032 -
309.1250 -302.9453 -308.3046 -306.3036 -310.8076 -308.3658

Has the environmental noise floor increased?

Question: How much noise is in the computational environment?

Assume $A_k = -310$ dB for $0 \leq k \leq N$

$$A_{\text{kdB}} = 20 \log_{10} A_k \quad \longrightarrow \quad A_k = 10^{\frac{A_{\text{kdB}}}{20}}$$

$$V_{\text{Noise,RMS}} \cong \bar{A} \sqrt{\frac{N}{2}} = 10^{-15.5} \sqrt{\frac{512}{2}} = 5.1 \cdot 10^{-15} \cong 5 \text{ fV} \quad N_p=31 \quad N=512$$

Assume $A_k = -310$ dB for $0 \leq k \leq N$ $N_p=97 \quad N=65536$

$$V_{\text{Noise,RMS}} \cong \bar{A} \sqrt{\frac{N}{2}} = 10^{-15.5} \sqrt{\frac{65536}{2}} = 57 \cdot 10^{-15} \cong 57 \text{ fV}$$

Note: **This computational environment is still very low even when N_p and N become large**

Considerations for Spectral Characterization

FFT Length

- FFT Length does not significantly affect the computational noise floor
- Although not shown here yet, FFT length does reduce the quantization noise floor coefficients

If we assume E_{QUANT} is fixed

$$E_{\text{QUANT}} \cong \sqrt{2^{n_{\text{DFT}}} \sum_{k=2} A_k^2}$$

If the A_k 's are constant and equal

$$E_{\text{QUANT}} \cong A_k 2^{n_{\text{DFT}}/2}$$

Solving for A_k , obtain

$$A_k \cong \frac{E_{\text{QUANT}}}{2^{n_{\text{DFT}}/2}}$$

If input is full-scale sinusoid with only amplitude quantization with n-bit res,

$$E_{\text{QUANT}} \cong \frac{X_{\text{LSB}}}{\sqrt{12}} = \frac{X_{\text{REF}}}{\sqrt{3} \bullet 2^{n+1}}$$

Considerations for Spectral Characterization

FFT Length

$$E_{\text{QUANT}} \cong \frac{X_{\text{LSB}}}{\sqrt{12}} = \frac{X_{\text{REF}}}{\sqrt{3} \bullet 2^{n+1}}$$

Substituting for E_{QUANT} , obtain



$$A_k \cong \frac{X_{\text{REF}}}{\sqrt{3} \bullet 2^{n+1} 2^{n_{\text{DFT}}/2}}$$

This value for A_k thus decreases with the length of the DFT window

Example: if $n=16$, $n_{\text{DFT}}=12$ (4096 pt transform), and $X_{\text{REF}}=1\text{V}$, then $A_k=6.9\text{E-}8\text{V}$ (-143dB),

(Note $A_k \gg$ computational noise for all practical n , n_{DFT})

Considerations for Spectral Characterization

- Tool Validation
- FFT Length
-  • Importance of Satisfying Hypothesis
 -  - NP is an integer
 - Band-limited excitation
- Windowing

Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

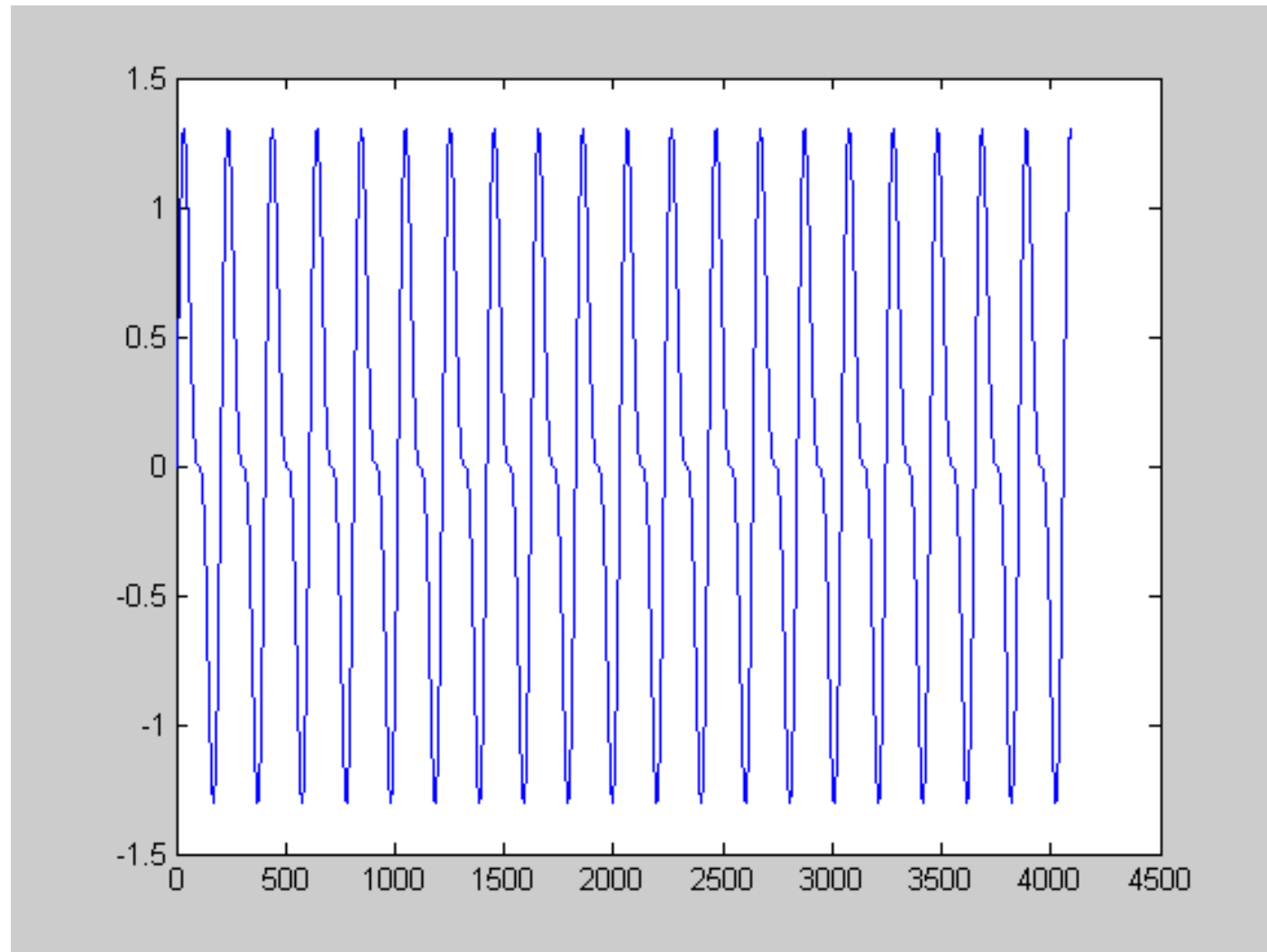
$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

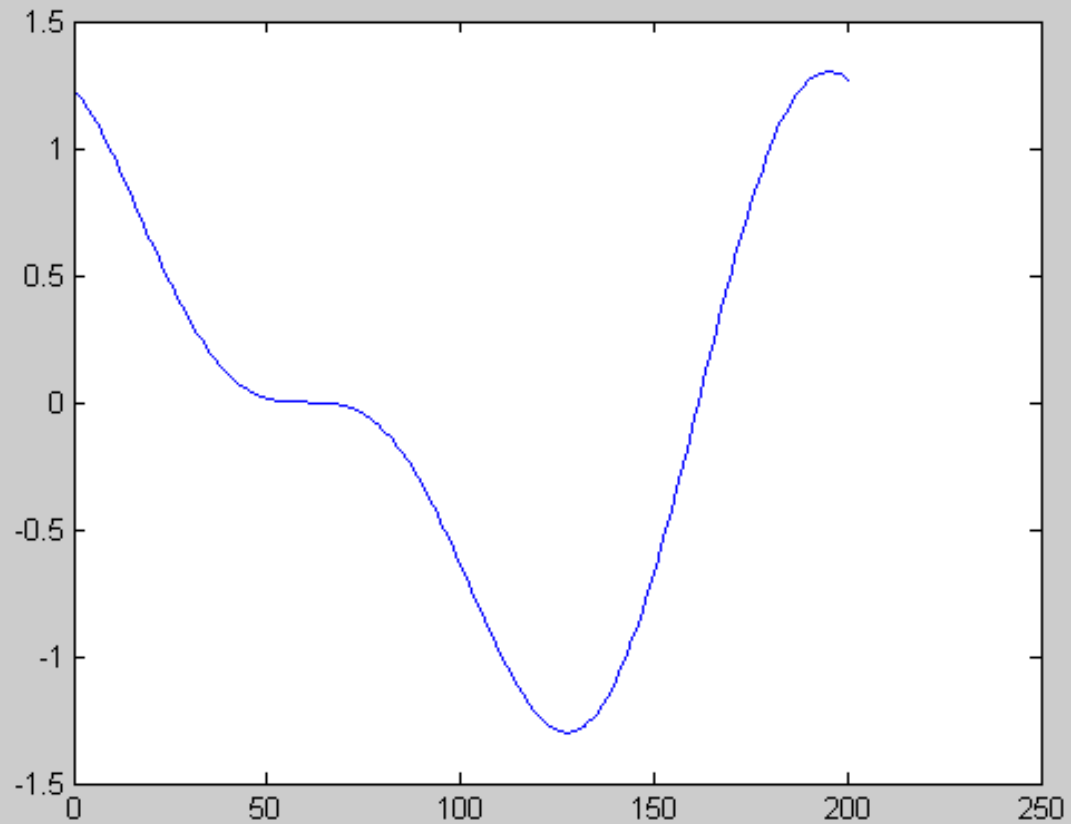
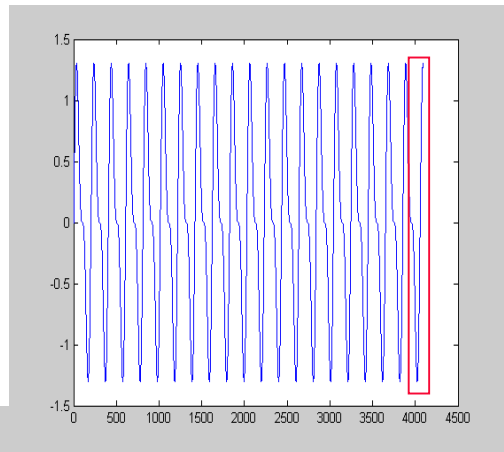
Consider $N_p=20.2$ $N=4096$

Recall $20\log_{10}(0.5)=-6.0205999$

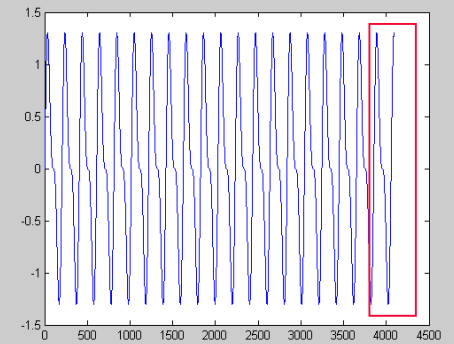
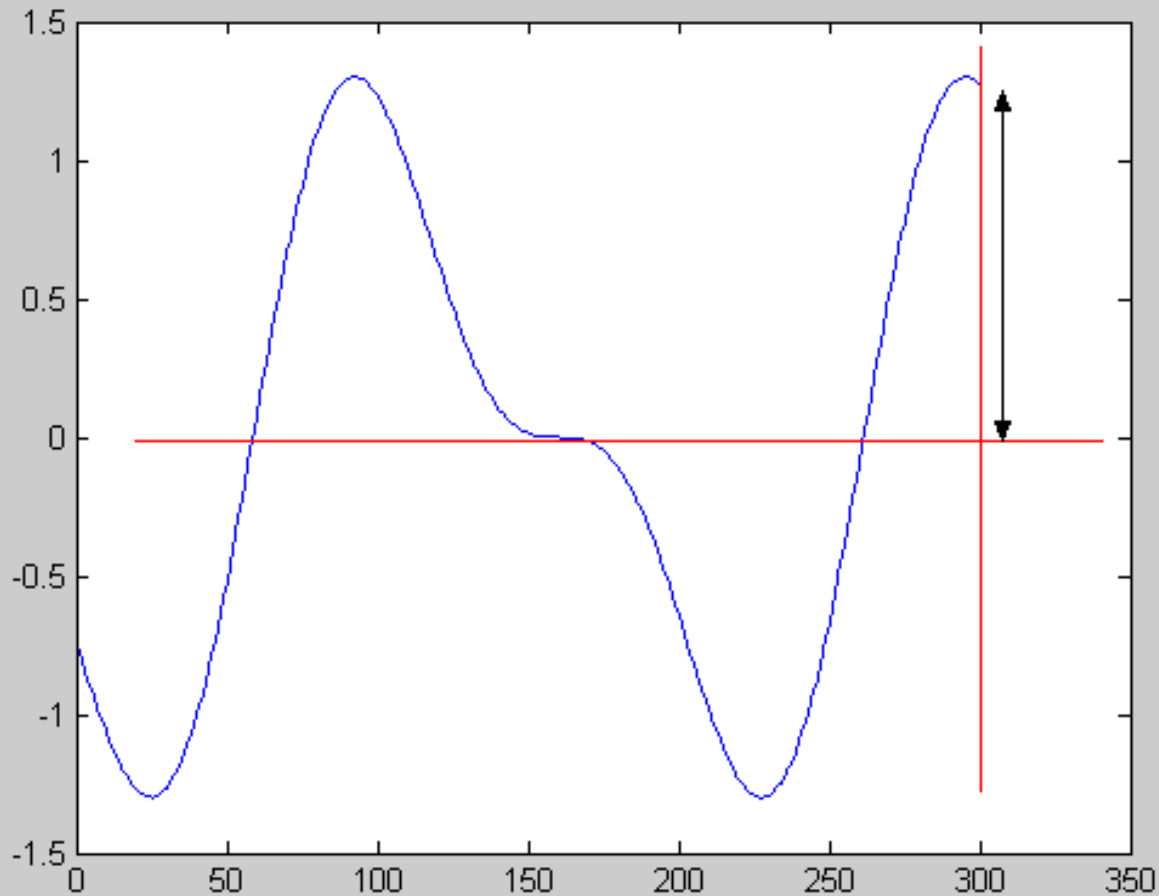
Input Waveform



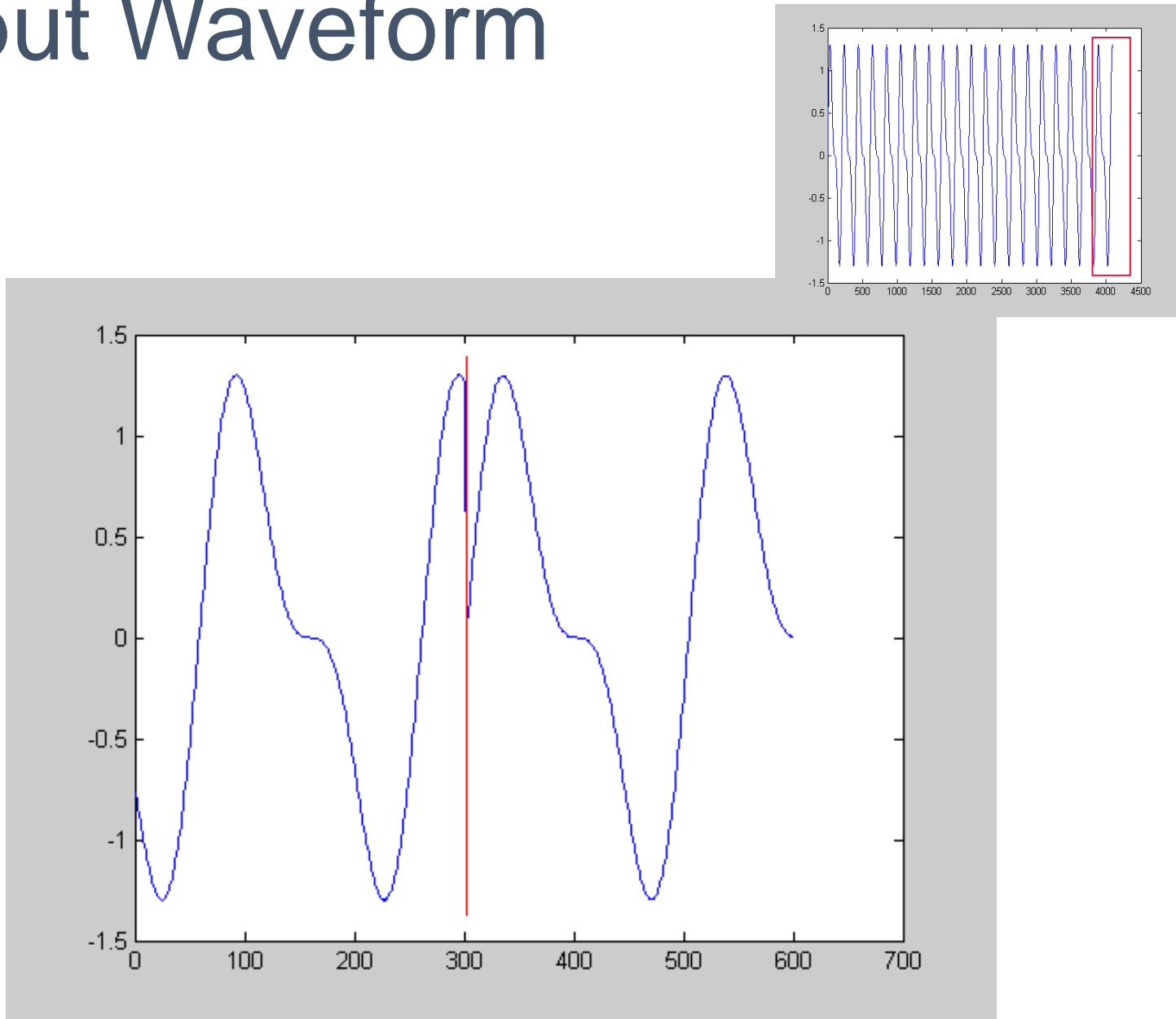
Input Waveform



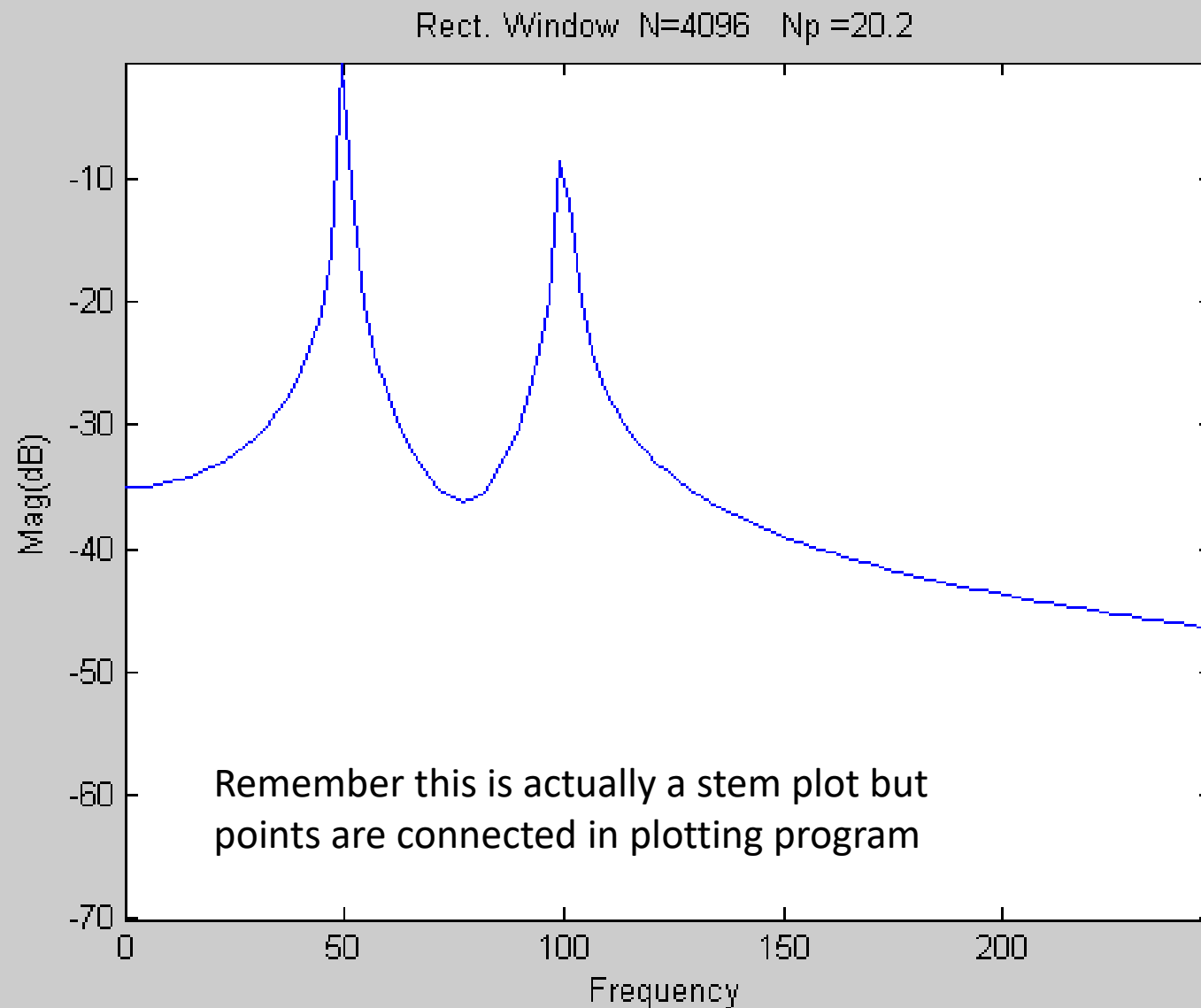
Input Waveform



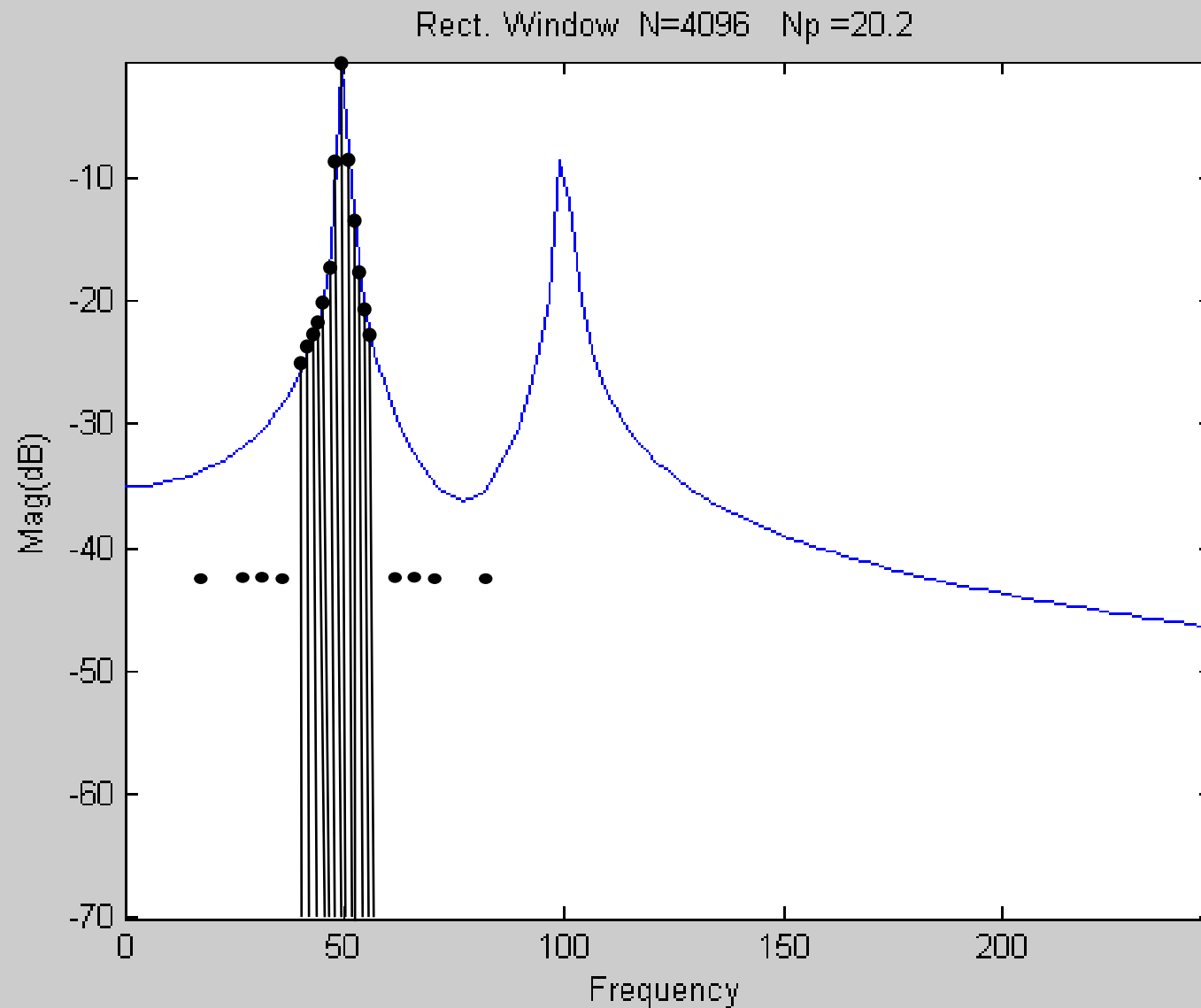
Input Waveform



Spectral Response



Spectral Response



Fundamental will appear at position $1+Np = 21$

Columns 1 through 7

-35.0366 -35.0125 -34.9400 -34.8182 -34.6458 -34.4208 -34.1403

Columns 8 through 14

-33.8005 -33.3963 -32.9206 -32.3642 -31.7144 -30.9535 -30.0563

Columns 15 through 21

-28.9855 -27.6830 -26.0523 -23.9155 -20.8888 -15.8561 **-0.5309**

Columns 22 through 28

-12.8167 -20.1124 -24.2085 -27.1229 -29.4104 -31.2957 -32.8782

Columns 29 through 35

-34.1902 -35.2163 -35.9043 -36.1838 -35.9965 -35.3255 -34.1946

Note there is a dramatic increase in the noise floor and a significant change in and spreading of the fundamental!!

k^{th} harmonic will appear at position $1+k \cdot Np$

Columns 36 through 42

-32.6350 -30.6397 -28.1125 -24.7689 -19.7626 -8.5639 -11.7825

Columns 43 through 49

-20.0158 -23.9648 -26.5412 -28.4370 -29.9279 -31.1519 -32.1874

Columns 50 through 56

-33.0833 -33.8720 -34.5759 -35.2113 -35.7902 -36.3218 -36.8133

Columns 57 through 63

-37.2703 -37.6974 -38.0984 -38.4762 -38.8336 -39.1725 -39.4949

Columns 64 through 70

-39.8024 -40.0963 -40.3778 -40.6479 -40.9076 -41.1576 -41.3987

k^{th} harmonic will appear at position $1+k \cdot N_p$

Columns 36 through 42

-32.6350 -30.6397 -28.1125 -24.7689 -19.7626 -8.5639 -11.7825

Columns 43 through 49

-20.0158 -23.9648 -26.5412 -28.4370 -29.9279 -31.1519 -32.1874

Columns 50 through 56

-33.0833 -33.8720 -34.5759 -35.2113 -35.7902 -36.3218 -36.8133

Columns 57 through 63

-37.2703 -37.6974 -38.0984 -38.4762 -38.8336 -39.1725 -39.4949

Columns 64 through 70

-39.8024 -40.0963 -40.3778 -40.6479 -40.9076 -41.1576 -41.3987

Observations

- Modest change in sampling window of 0.2 out of 20 periods (1%) results in a big error in both fundamental and harmonic
- More importantly, dramatic raise in the “noise floor” !!! (from over -300dB to only -12dB)

Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

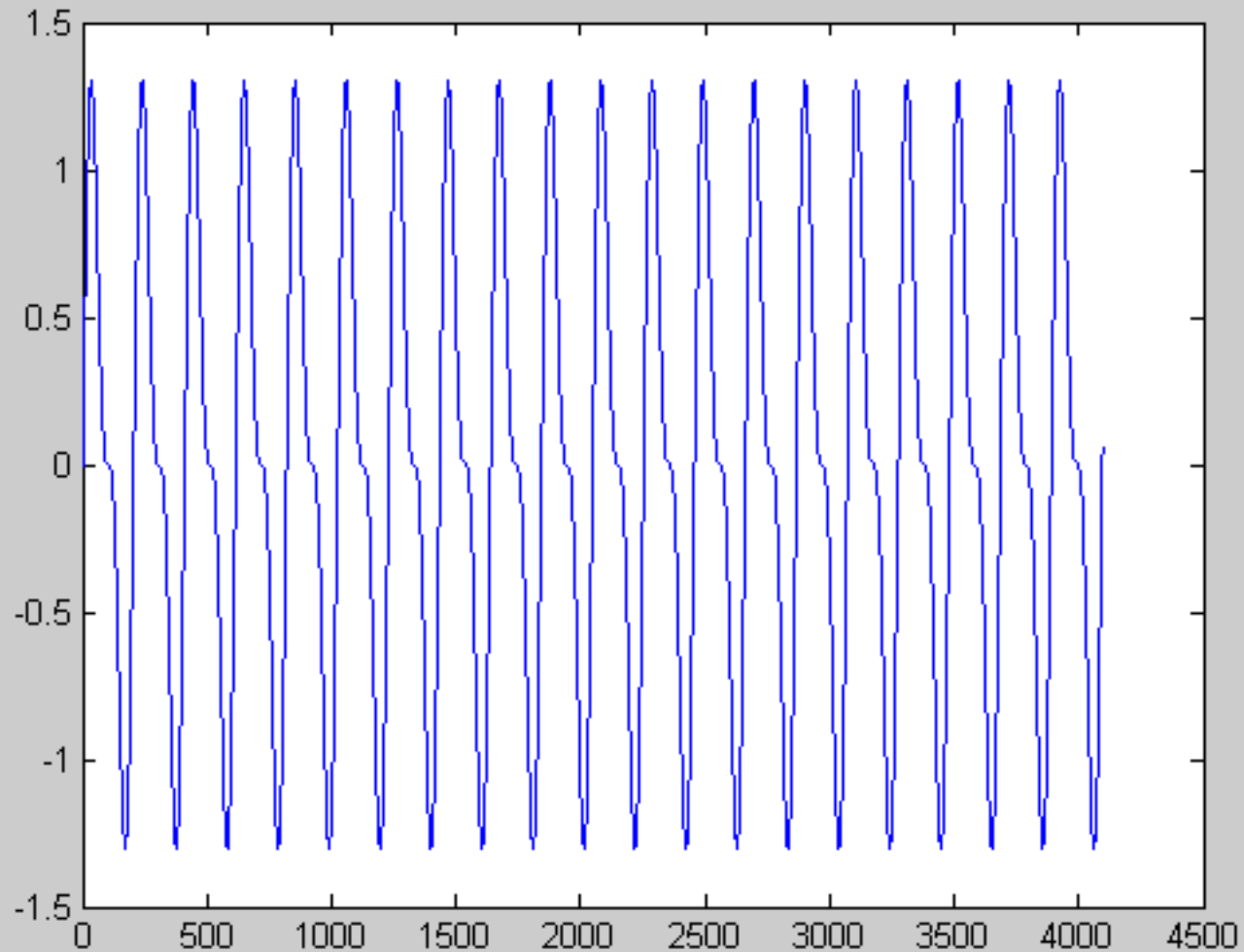
$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

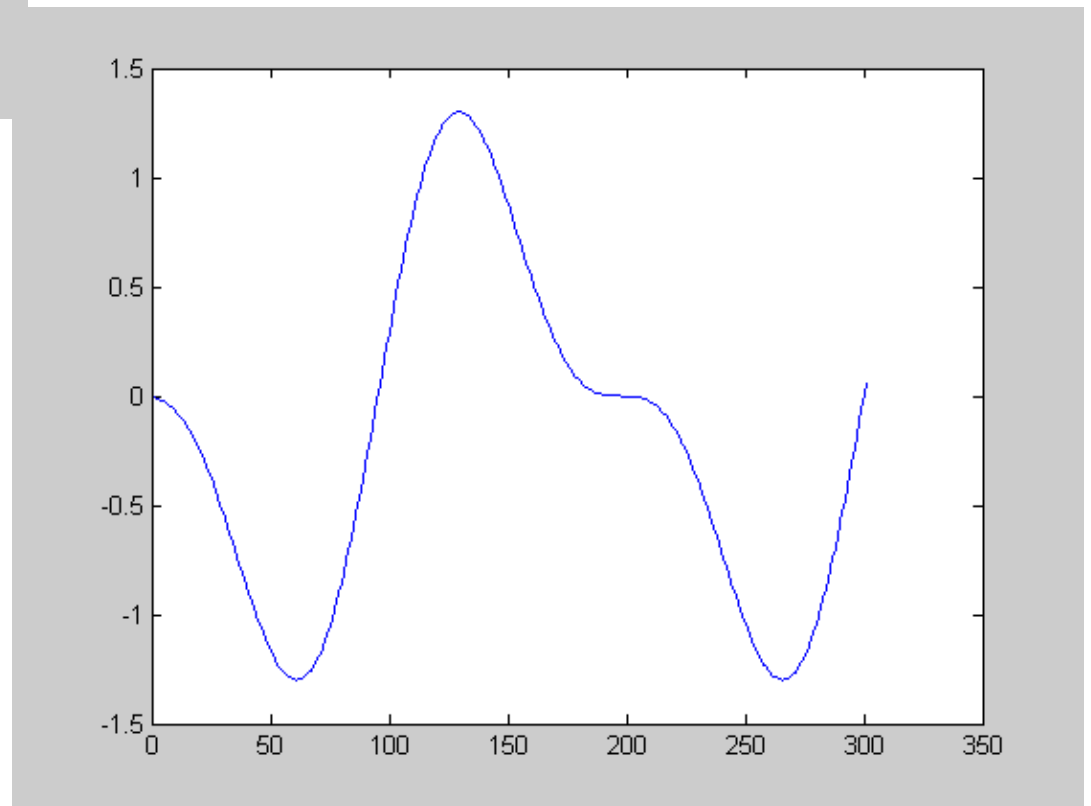
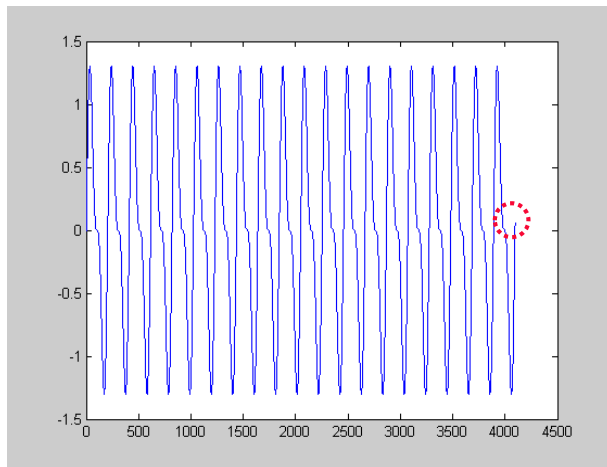
Consider $N_p=20.01$ $N=4096$

Deviation from hypothesis is .05% of the sampling window

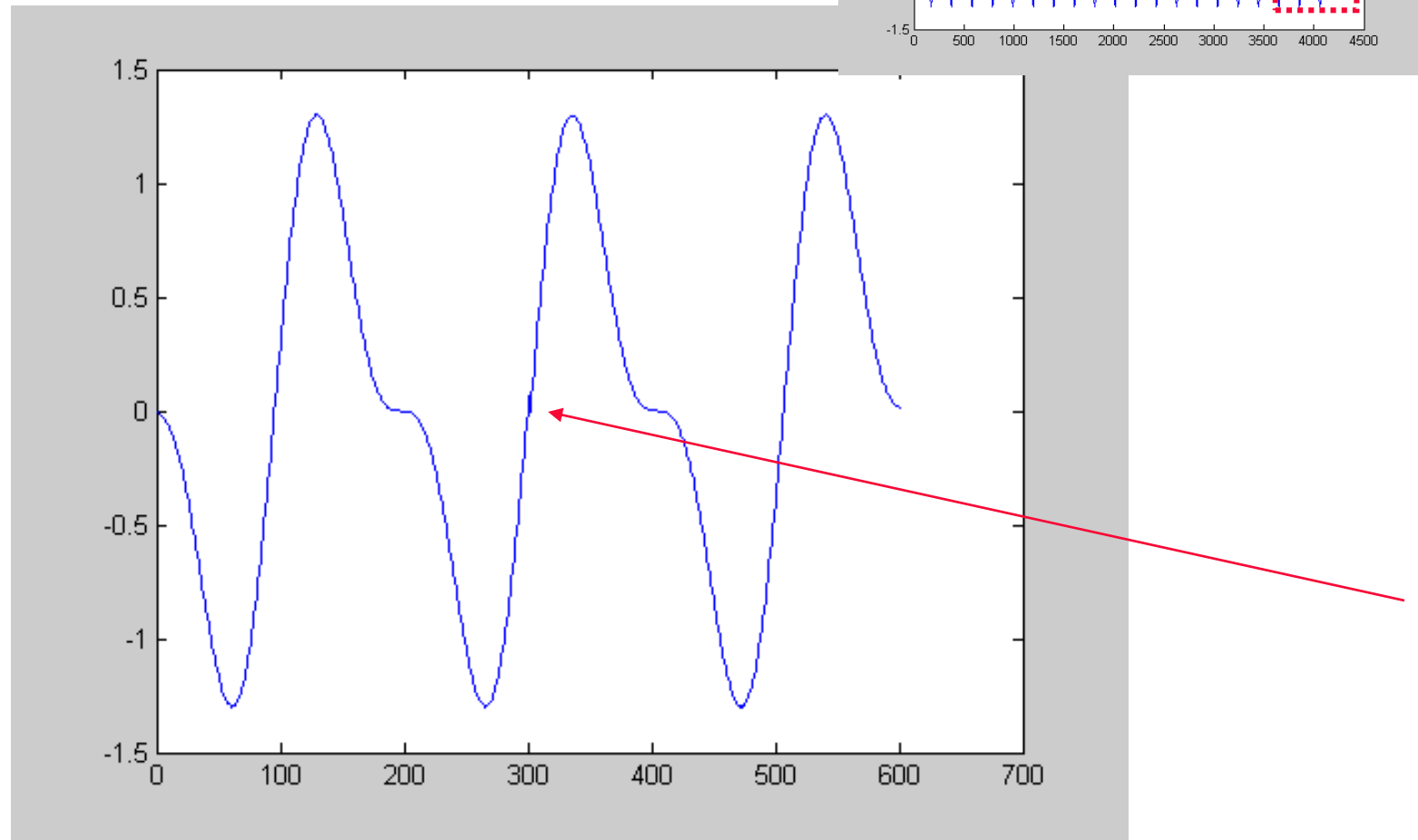
Input Waveform



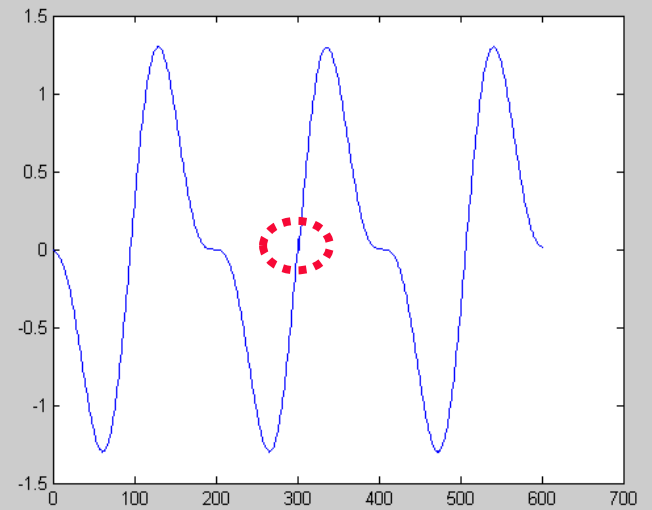
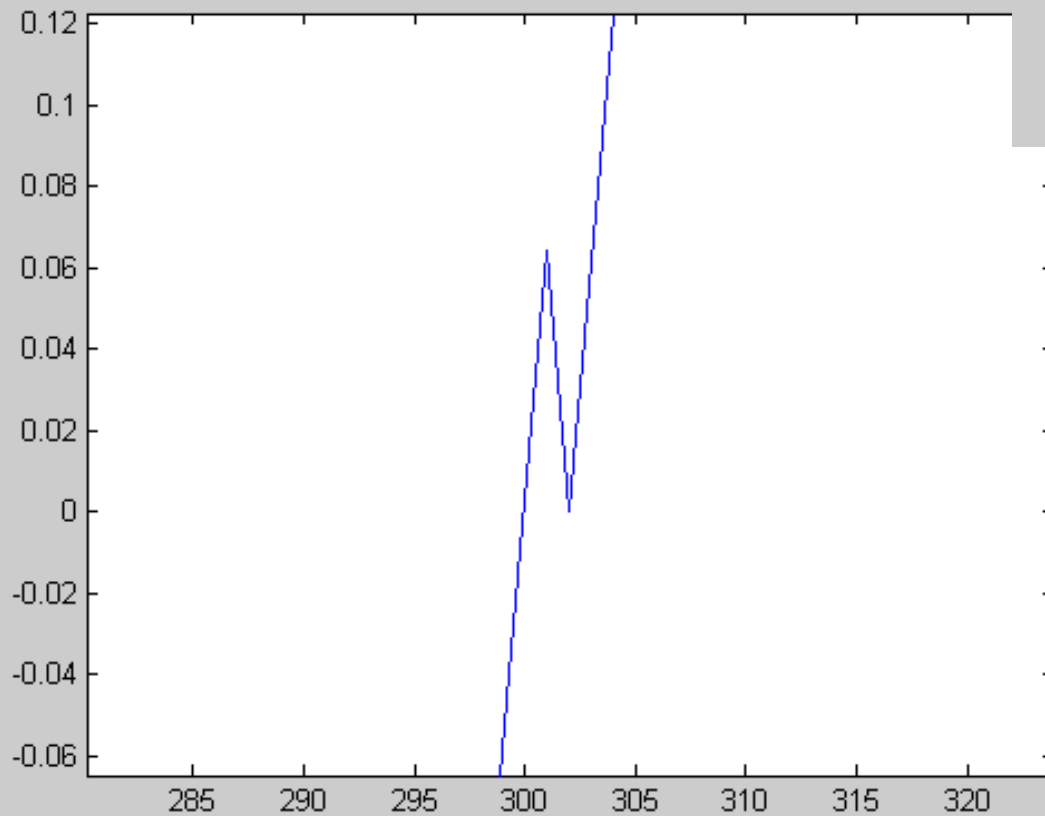
Input Waveform



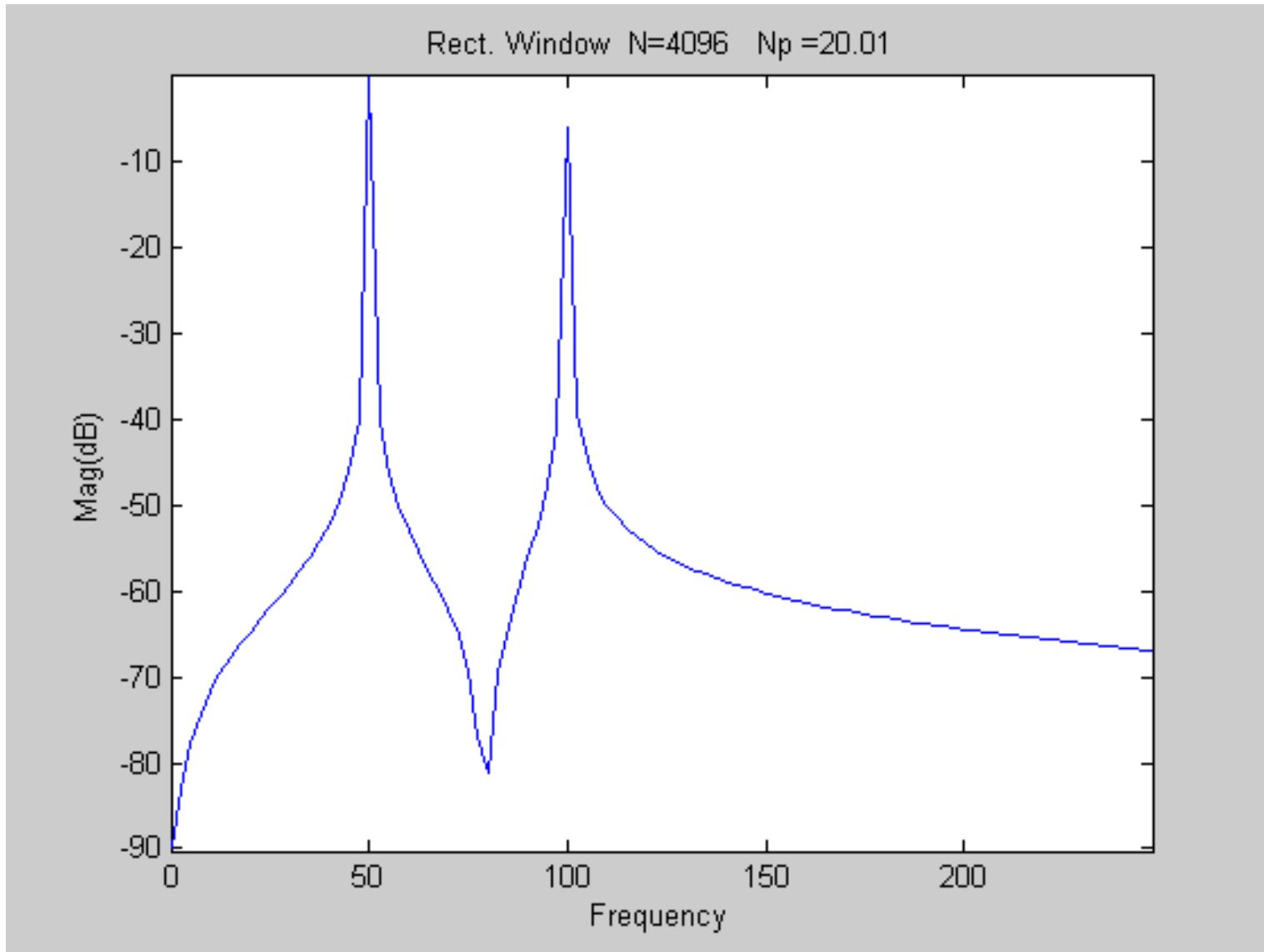
Input Waveform



Input Waveform



Spectral Response with Non-Coherent Sampling



(zoomed in around fundamental)

Fundamental will appear at position $1+Np = 21$

Columns 1 through 7

-89.8679 -83.0583 -77.7239 -74.2607 -71.6830 -69.5948 -67.8044

Columns 8 through 14

-66.2037 -64.7240 -63.3167 -61.9435 -60.5707 -59.1642 -57.6859

Columns 15 through 21

-56.0866 -54.2966 -52.2035 -49.6015 -46.0326 -40.0441 -0.0007

Columns 22 through 28

-40.0162 -46.2516 -50.0399 -52.8973 -55.3185 -57.5543 -59.7864

Columns 29 through 35

-62.2078 -65.1175 -69.1845 -76.9560 -81.1539 -69.6230 -64.0636

k^{th} harmonic will appear at position $1+k \cdot Np$

Columns 36 through 42

-59.9172 -56.1859 -52.3380 -47.7624 -40.9389 -6.0401 -39.2033

Observations

- Modest change in sampling window of 0.01 out of 20 periods (.05%) still results in a modest error in both fundamental and harmonic
- More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -40dB)
- Errors at about the 6-bit level !

Example

WLOG assume $f_{\text{SIG}}=50\text{Hz}$

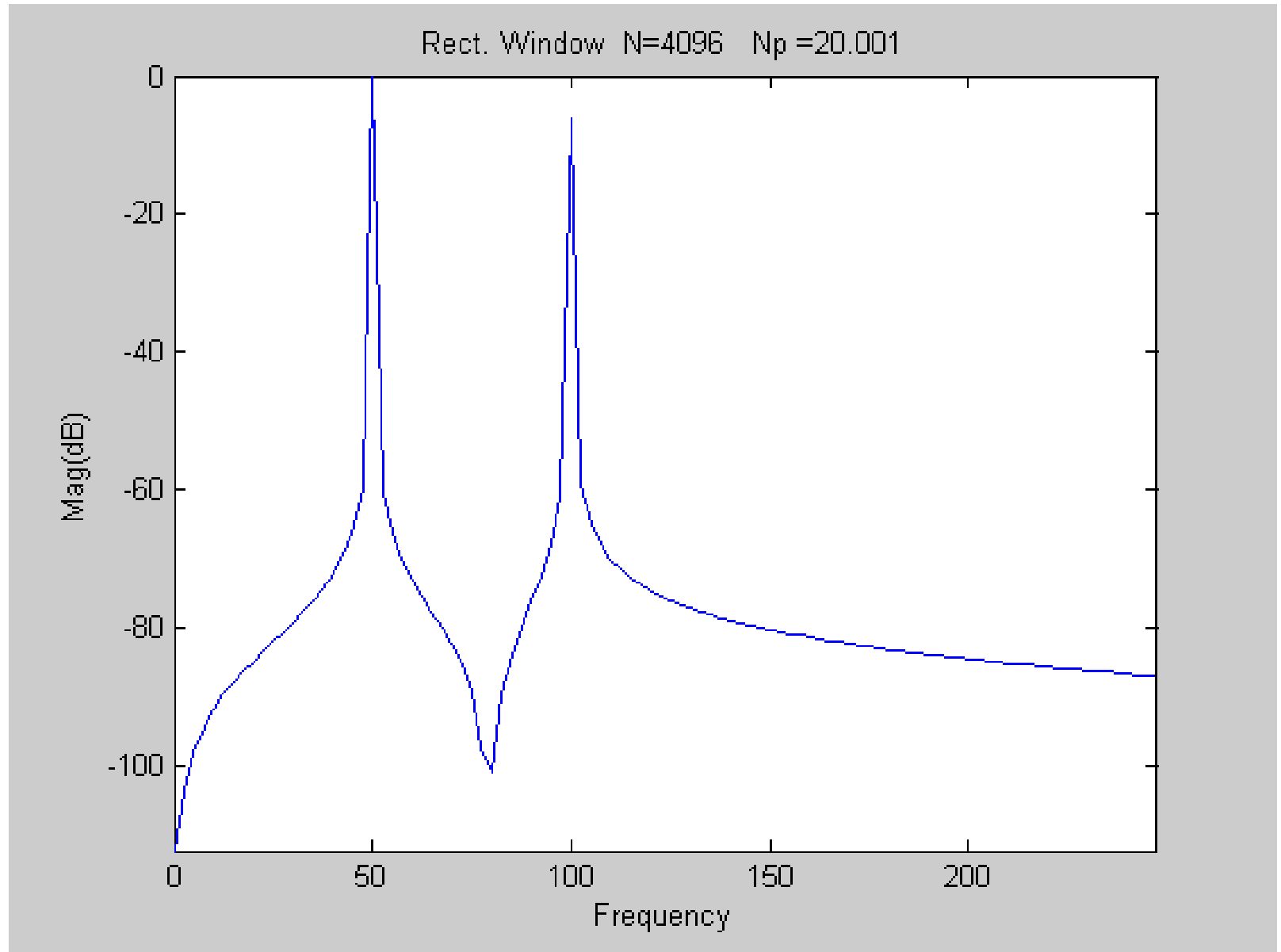
$$V_{\text{IN}} = \sin(\omega t) + 0.5 \sin(2\omega t)$$

$$\omega = 2\pi f_{\text{SIG}}$$

Consider $N_p=20.001$ $N=4096$

Deviation from hypothesis is .005% of the sampling window

Spectral Response with Non-coherent Sampling



(zoomed in around fundamental)

Fundamental will appear at position $1+Np = 21$

Columns 1 through 7

-112.2531 -103.4507 -97.8283 -94.3021 -91.7015 -89.6024 -87.8059

Columns 8 through 14

-86.2014 -84.7190 -83.3097 -81.9349 -80.5605 -79.1526 -77.6726

Columns 15 through 21

-76.0714 -74.2787 -72.1818 -69.5735 -65.9919 -59.9650 0.0001

Columns 22 through 28

-60.0947 -66.2917 -70.0681 -72.9207 -75.3402 -77.5767 -79.8121

Columns 29 through 35

-82.2405 -85.1651 -89.2710 -97.2462 -101.0487 -89.5195 -83.9851

k^{th} harmonic will appear at position $1+k \cdot N_p$

Columns 36 through 42

-79.8472 -76.1160 -72.2601 -67.6621 -60.7642 -6.0220 -59.3448

Columns 43 through 49

-64.8177 -67.8520 -69.9156 -71.4625 -72.6918 -73.7078 -74.5718

Columns 50 through 56

-75.3225 -75.9857 -76.5796 -77.1173 -77.6087 -78.0613 -78.4809

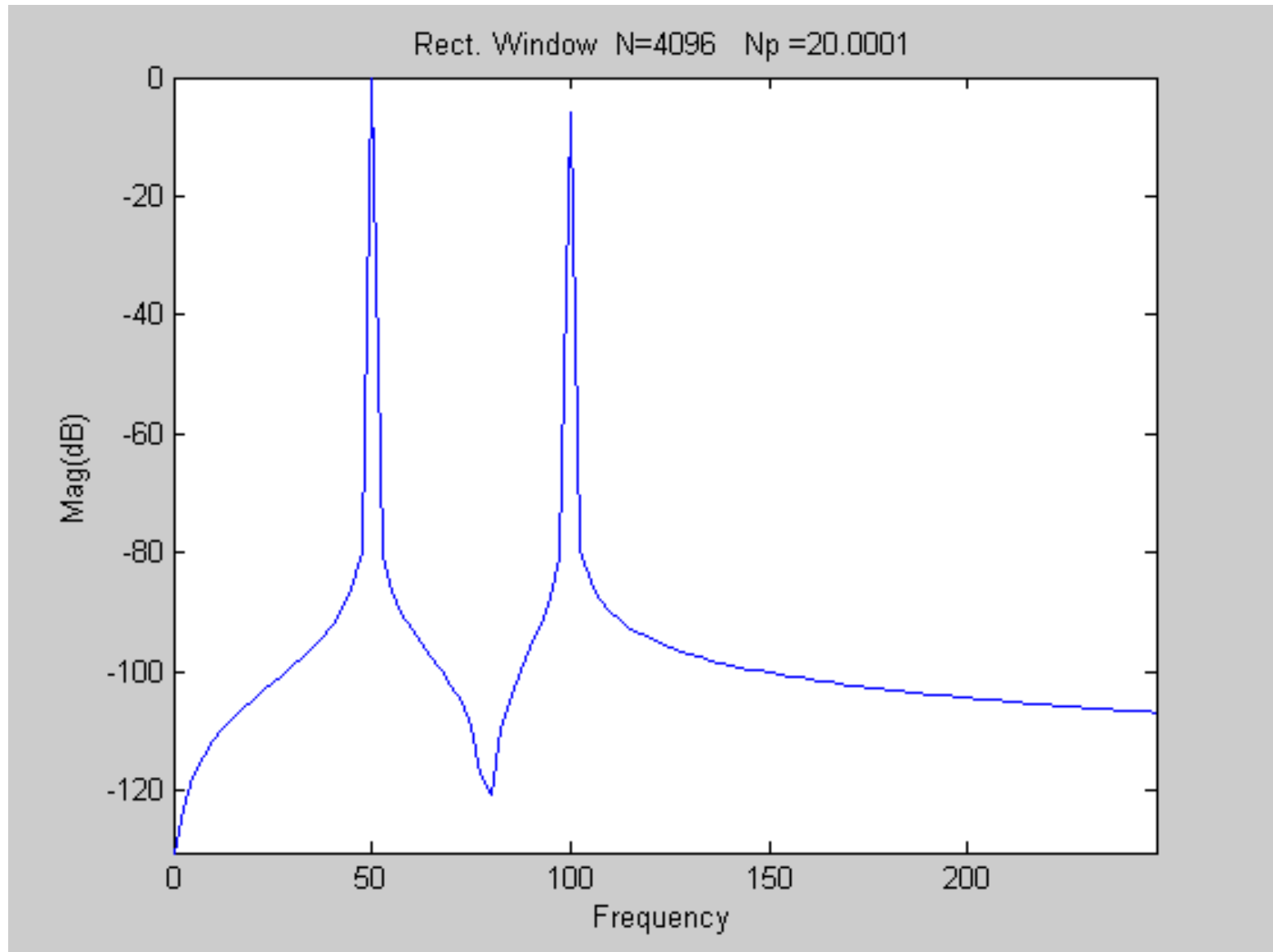
Columns 57 through 63

-78.8721 -79.2387 -79.5837 -79.9096 -80.2186 -80.5125 -80.7927

Observations

- Modest change in sampling window of 0.01 out of 20 periods (.005%) results in a small error in both fundamental and harmonic
- More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -60dB)
- Errors at about the 10-bit level !

Spectral Response with Non-coherent sampling



(zoomed in around fundamental)

Fundamental will appear at position $1+Np = 21$

Columns 1 through 7

-130.4427 -123.1634 -117.7467 -114.2649 -111.6804 -109.5888 -107.7965

Columns 8 through 14

-106.1944 -104.7137 -103.3055 -101.9314 -100.5575 -99.1499 -97.6702

Columns 15 through 21

-96.0691 -94.2764 -92.1793 -89.5706 -85.9878 -79.9571 0.0000

Columns 22 through 28

-80.1027 -86.2959 -90.0712 -92.9232 -95.3425 -97.5788 -99.8141

Columns 29 through 35

-102.2424 -105.1665 -109.2693 -117.2013 -120.8396 -109.4934 -103.9724

k^{th} harmonic will appear at position $1+k \cdot N_p$

Columns 36 through 42

-99.8382 -96.1082 -92.2521 -87.6522 -80.7470 -6.0207 -79.3595

Columns 43 through 49

-84.8247 -87.8566 -89.9190 -91.4652 -92.6940 -93.7098 -94.5736

Columns 50 through 56

-95.3241 -95.9872 -96.5810 -97.1187 -97.6100 -98.0625 -98.4821

Columns 57 through 63

-98.8732 -99.2398 -99.5847 -99.9107 -100.2197 -100.5135 -100.7937

Columns 64 through 70

Observations

- Modest change in sampling window of 0.001 out of 20 periods (.0005%) results in a small error in both fundamental and harmonic
- More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -80dB)
- Errors at about the 13-bit level !

Considerations for Spectral Characterization

- Tool Validation

- FFT Length

-  • Importance of Satisfying Hypothesis

 - NP is an integer

 -  - Band-limited excitation

- Windowing



Stay Safe and Stay Healthy !

End of Lecture 5